

# Maple: Derivatives and Critical Points

## Introduction

We know that Maple is able to carry out symbolic algebraic calculations quite easily. Due to this fact Maple is an ideal package for solving symbolic calculations relating to calculus. In this chapter we shall demonstrate how Maple can be used to find the derivative of a function, determine the critical points of the function, if they exist, and assist you in classifying the critical points of the function.

## Calculating the Derivative of a Function

Define the function `f` as

```
> f := x^3*(x^2-1);
```

$$f := x^3(x^2 - 1)$$

The derivative of a function is computed using the `diff` command. To find out more about how to correctly use the `diff` command we can look up the Maple help file associated with it. To look up the Maple help file for a command type a question mark following immediately (no space) by the command and hit **Enter**. For example

```
> ?diff
```

The help file will appear in a new window showing you the correct command syntax as well as several examples of how to use the command.

To calculate the first derivative of a function using the `diff` command we need to identify what function we are differentiating and we also need to specify what variable we are differentiating with respect to. If we wish to differentiate the function `f`, defined above, with respect to `x` then we would type

```
> diff(f,x);
```

$$3x^2(x^2 - 1) + 2x^4$$

There are two ways that we can compute the second derivative of the function  $f$ . The first approach is to assign a name, for example  $g$ , to the output from the first derivative and then to calculate `diff(g,x)` as shown below;

```
> g:=diff(f,x);
```

$$g := 3x^2(x^2 - 1) + 2x^4$$

```
> diff(g,x);
```

$$6x(x^2 - 1) + 14x^3$$

The second approach is more direct - in this approach we simply instruct Maple to differentiate the derivative of the function  $f$  twice to give us the second derivative.

```
> diff(f,x,x);
```

$$6x(x^2 - 1) + 14x^3$$

We can compute any order of derivative that we want. The maple command to compute the 4<sup>th</sup> derivative is as follows;

```
> diff(f,x$4);
```

$$120x$$

If you compare the output to the output that you would receive from the command `diff(f,x,x,x,x)`; you will see that they are exactly the same. The approach utilising `x$4` is quicker due to less writing but is also more syntactically correct as it is easier to change to higher order derivatives than in the other approach.

If you are given a function of several variable then Maple can compute the derivatives quite easily. The key point when dealing with a function of several variables is specifying the variable that you are differentiating with respect to.

```
> h := 3*x*y^2*(x^2-4*y);
```

$$h := 3xy^2(x^2 - 4y)$$

```
> diff(h,x);
```

$$3y^2(x^2 - 4y) + 6x^2y^2$$

```
> diff(h,y);
```

$$6xy(x^2 - 4y) - 12xy^2$$

## Calculating the Critical Points

A **critical point** is a point in the domain of a function where the function ceases to be differentiable. This means that the slope of the function evaluated at this point is equal to zero. Before proceeding and attempting to calculate the critical points of a function it might be beneficial if we could first plot the function so that we can estimate approximately the value(s) of the critical point(s). We shall use the function **f** that we defined in the previous section,

```
> f := x^3*(x^2-1);
```

$$f := x^3(x^2 - 1)$$

To plot a function in Maple we use the **plot** command. The plot command has two arguments: the first is the function that we are plotting and the second is the interval over which we will plot the function. Therefore plotting the function **f** in the interval  $[-1, +1]$  is done as follows;

```
> plot(f,x=-1..1);
```

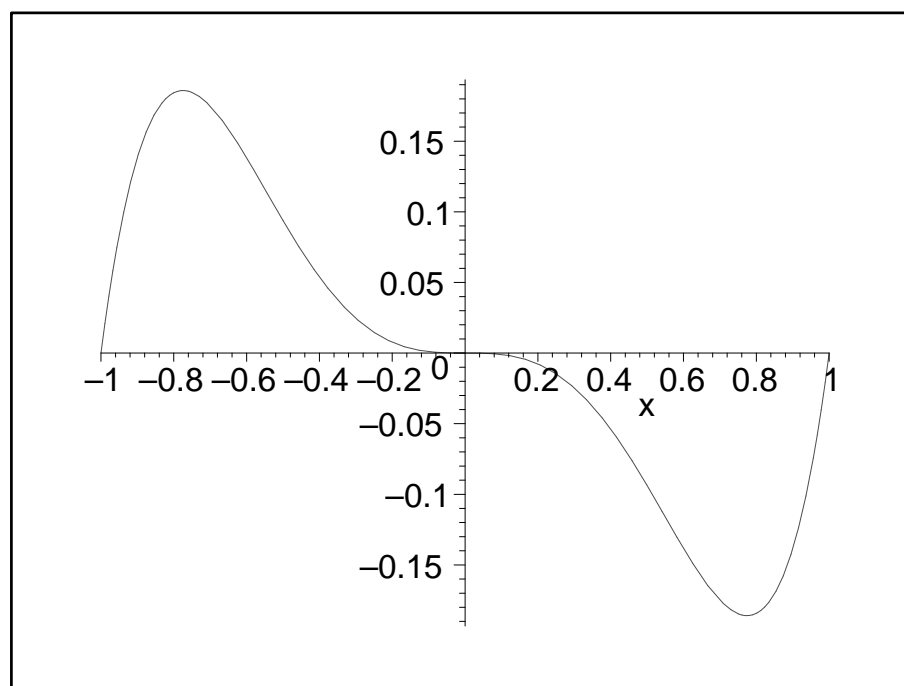


Figure 1: Plot of  $y = x^3(x^2 - 1)$  between -1 and +1.

Inspection of the graph would seem to suggest that there are three points where the tangent line would be parallel to the  $x$ -axis:

1. A point in the interval  $(-1, 0)$ , which is near  $x = -0.75$ .

2. A point near  $x=0$ .
3. A point in the interval  $(0, 1)$ , which is near  $x=0.75$ .

To calculate the critical points we differentiate the function and then solve it equal to zero. We shall assign the label `df` to the derivative of the function `f`.

```
> df := diff(f,x);  

$$df := 3x^2(x^2 - 1) + 2x^4$$

```

Solving `df` equal to zero using the `solve` command yields the critical points. Maple, by default, gives the critical points as fractions. If you wish to convert the answers into floating-point form we can use the command `evalf` to do this.

```
> cp := evalf(solve(df=0,x));  

$$cp := 0., 0., 0.7745966692, -0.7745966692$$

```

Maple tells us that there are four critical points but by inspection it is clear that two of these critical points, located at the origin, are the same. We can calculate the critical value associated with each of the critical points by substituting each of the critical points into the original equation for `f`. Recall that `cp` refers to all four of the answers provided by Maple and to access each of the individual critical points we use square brackets, e.g. `cp[1]` would return the value of the first critical point listed in `cp` i.e. 0.

```
> evalf( subs(x=cp[1], f) );  

$$-0.$$

```

Therefore the coordinate of the first critical point is  $(0, 0)$ . `cp[2]` will give the same result and so we no longer need to consider it. The critical values associated with `cp[3]` and `cp[4]` are calculated in a similar fashion.

```
> evalf( subs(x=cp[3], f) );  

$$-0.1859032006$$
  
> evalf( subs(x=cp[4], f) );  

$$0.1859032006$$

```

Therefore the co-ordinates of the final two critical points are

$$(0.7745966692, -0.1859032006) \quad \text{and} \quad (-0.7745966692, 0.1859032006).$$

To check if the critical points are local maximum, minimum or points of inflection we carry out the second derivative test. The second derivative tests states the following:

If the function  $f$  is twice differentiable in the neighbourhood of a fixed point  $x$ , i.e.  $f'(x) = 0$  then

- if  $f''(x) < 0$  then  $f$  has a local maximum point at  $x$ ,
- if  $f''(x) > 0$  then  $f$  has a local minimum point at  $x$ ,
- if  $f''(x) = 0$  then the second derivative test is inconclusive about the point  $x$  - it could be a point of inflection but it may not be.

If  $f''(x) = 0$  then a simple way to test if the critical point is a point of inflection is to evaluate  $f''$  at two points in close proximity to the critical point, one on either side of the critical point, and see if they yield different signs. If they do then the critical point that you are analysing is a point of inflection. Therefore we need to calculate the second derivative of the function  $f$  and then evaluate the value of the second derivative at each of the critical points.

```
> d2f := diff(df,x);
```

$$d2f := 6x(x^2 - 1) + 14x^3$$

Evaluate the second derivative at the critical point `cp[1]`, i.e.  $x = 0$ .

```
> evalf(subs(x=cp[1],d2f));
```

0.

This test is inconclusive and so we need to evaluate  $f''$  at two points in close proximity to  $x = 0$ . The two points we will look at shall be  $x = -0.15$  and  $x = +0.15$ .

```
> evalf(subs(x=0.15,d2f));
```

-0.832500

```
> evalf(subs(x=-0.15,d2f));
```

0.832500

Due to the fact that the sign of these two values is different the point  $x = 0$  is in fact a point of inflection.

Evaluate the second derivative at the critical point `cp[3]`, i.e.  $x = 0.7745966692$ .

```
> evalf(subs(x=cp[3],d2f));
```

4.647580014

The second derivative is positive which implies that the critical point  $x = 0.7745966692$  is a local minimum. Finally evaluate the second derivative at the critical point `cp[4]`, i.e.  $x = -0.7745966692$ .

```
> evalf(subs(x=cp[4],d2f));
```

-4.647580014

The second derivative is negative which implies that the critical point  $x = -0.7745966692$  is a local maximum. We have thus calculated and classified the critical points associated with a given function.