Converging Series

Aim

To demonstrate a method for checking if a series is convergent or not.

Learning Outcomes

At the end of this section you will:

- Know how to check if a specific series is convergent or not,
- Know how to use the ratio test for convergence.

In the previous section we established that the sum to infinity of a series is defined as the limit of S_N as N tends to infinity.

When a series has a limit (i.e. $\lim_{N\to\infty} S_N$ is finite (tends towards a value)), we say that the series converges or is *convergent*. When a series has no limit (i.e. tends to infinity), we say that the series diverges or is *divergent*.

To check if a series converges or not we use the following very simple test.

Let $\sum a_n$ be a given series such that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p.$$

If $p < 1$, the series is *convergent*.
If $p > 1$, the series is *divergent*.
If $p = 1$, the test is inconclusive.

The above test is known as the ratio test. The ratio test is generally used to test for convergence or divergence a series in which

1. the variable (generally n) appears in factorial form, $a_n = \dfrac{3n!}{n+1}$
2. the variable appears as a power, $a_n = \dfrac{2^n}{n^2}$

Example 1

Test if the following series is convergent or not.

$$a_n = \frac{x^{n-1}}{(n-1)!}$$

We know

$$a_n = \frac{x^{n-1}}{(n-1)!} \quad \text{and} \quad a_{n+1} = \frac{x^n}{(n)!}$$

$$\begin{aligned} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^n}{n!} / \frac{x^{n-1}}{(n-1)!}\right| \\ &= |x| \frac{(n-1)!}{n!} \\ &= |x| \frac{(n-1)!}{(n-1)!n} = \frac{|x|}{n} \end{aligned}$$

But x is fixed and so

$$\lim_{n\to\infty} \frac{|x|}{n} = 0 < 1.$$

Therefore the series is convergent.

Example 2

Test if the following series is convergent or not.

$$a_n = \frac{2^n}{n^2}$$

We know

$$a_n = \frac{2^n}{n^2} \quad \text{and} \quad a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$$

$$\begin{aligned} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{2^{n+1}}{(n+1)^2} / \frac{2^n}{n^2}\right| \\ &= \left|\frac{2^{n+1}}{(n+1)^2} . \frac{n^2}{2^n}\right| \\ &= \left|\frac{2n^2}{(n+1)^2}\right| . \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{2n^2}{(n+1)^2}\right| = 2 > 1.$$

Therefore this series is divergent.

Related Reading

Gersting, J.L. 2007. *Mathematical Structures for Computer Science.* 6^{th} Edition. Freeman & Company.

Morris, O.D. 1992. *Text & Tests 4.* The Celtic Press.