Converging & Diverging Sequences

Aim

To define what is meant by saying that a sequence converges or diverges.

Learning Outcomes

At the end of this section you will:

- Understand what a converging sequence is,
- Understand what a diverging sequence is,
- Know how to check if a sequence is converging or diverging.

Look at the following sequence

$$a_n = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}$$

The first few terms are $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ It is obvious that the terms in this sequence tend to approach 1 as n becomes larger and larger. Therefore it is possible to write

$$\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} \to 1 \qquad \text{as } n \to \infty$$

We can say that "1" is the *limit* of the above sequence as n tends to infinity. If a sequence has a limit, we say the sequence is *convergent*, and that the sequence converges to the limit. Otherwise, the sequence is *divergent*.

In order for a given sequence to converge to a limit:

$$\{a_n\}_{n=1}^{\infty} \to L \qquad (L = \text{some number})$$

what we really mean is

$$|a_n - L| \to 0 \quad \text{as} \quad n \to \infty$$

Not all sequences have a limit. Take for example any unbounded sequence. If a sequence has a limit it is the only limit there is for it. Filling values into the sequence and "guessing" that the sequence is bounded does not prove that the sequence has a limit. To find the limit of a sequence, if it exists, we must calculate

$$\lim_{n\to\infty} a_n$$

Example 1

Calculate the limit of the following sequence.

$$a_n = \frac{x^2+7}{x^2}$$

To take the limit of this sequence, or any sequence of this form, we divide each term in the sequence by the highest power of "x" that appears in the denominator and then take the limit,

$$\lim_{x\to\infty} \frac{x^2+7}{x^2} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{7}{x^2}}{\frac{x^2}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{7}{x^2}}{1} = \frac{1+\frac{7}{\infty}}{1} = \frac{1+0}{1} = 1$$

Therefore the limit of this sequence is 1. This is clearly a converging sequence.

Example 2

Calculate the limit of the following sequence.

$$a_n = \frac{x^2+3}{2x-2}$$

To take the limit of this sequence we again divide each term in the sequence by the highest power of "x" that appears in the denominator and then take the limit,

$$\lim_{x\to\infty} \frac{x^2+3}{2x-2} = \lim_{x\to\infty} \frac{\frac{x^2}{x}+\frac{3}{x}}{\frac{2x}{x}-\frac{2}{x}} = \lim_{x\to\infty} \frac{x+\frac{3}{x}}{2-\frac{2}{x}} = \frac{\infty+\frac{3}{\infty}}{2-\frac{2}{\infty}} = \frac{\infty+0}{2+0} = \frac{\infty}{2} = \infty$$

Therefore the limit of this sequence is ∞. This is clearly a diverging sequence.

Related Reading

Gersting, J.L. 2007. *Mathematical Structures for Computer Science.* 6^{th} Edition. Freeman & Company.

Morris, O.D. 1992. *Text & Tests 4.* The Celtic Press.