Computer Application of Series

Aim

To demonstrate an application of series when working with computers.

Learning Outcomes

At the end of this section you will:

- Be able to identify a possible application of series when working with computers,
- Know a possible method for representing roots on a computer.

This section is a general application of both the areas of sequences and series. Think about the following question - How are square roots computed on a computer ??

Here is one possible way. Define a sequence by a recursive rule. Take for example $\sqrt{2}$.

$$\begin{aligned} a_1 &= 1, \\ a_{n+1} &= \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \end{aligned}$$

Using this rule we can see that

$$\begin{aligned} a_2 &= a_{1+1} = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) \\ \therefore a_2 &= \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} \\ a_3 &= a_{2+1} = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) \\ \therefore a_3 &= \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) \backsimeq 1.4166 \quad \text{etc.} \end{aligned}$$

Assuming $\{a_n\}$ converges (this means $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$),

$$\Rightarrow L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{2}\left(a_n + \frac{2}{a_n}\right)\right)$$

Using the elementary properties of sequences, we know

$$\lim_{n\to\infty}\left(\frac{1}{2}\left(a_n + \frac{2}{a_n}\right)\right) = \lim_{n\to\infty}\left(\frac{1}{2}a_n\right) + \lim_{n\to\infty}\frac{1}{2}\left(\frac{2}{a_n}\right)$$

$$\begin{aligned}
\therefore L &= \frac{1}{2}\lim_{n\to\infty} a_n + \lim_{n\to\infty}\left(\frac{1}{a_n}\right) \\
L &= \frac{1}{2}L + \frac{1}{L} \Rightarrow \frac{1}{2}L = \frac{1}{L} \\
\Rightarrow \frac{1}{2}L^2 &= 1 \\
\Rightarrow L^2 &= 2 \Rightarrow L = \sqrt{2}
\end{aligned}$$

So the limit of the sequence is $\sqrt{2}$.

This method could be used to compute $\sqrt{2}$ by calculating a_5, as an approximation.

This approach can easily be adjusted to deal with other roots. If we had defined

$$\begin{aligned}
a_1 &= 1, \\
a_{n+1} &= \frac{1}{2}\left(a_n + \frac{p}{a_n}\right)
\end{aligned}$$

where p is any number ≥ 0, then this new sequence converges to $\sqrt{p}$.

Related Reading

Gersting, J.L. 2007. *Mathematical Structures for Computer Science.* 6^{th} Edition. Freeman & Company.