Bounded Sequence

Aim

To define what a bounded sequence is.

Learning Outcomes

At the end of this section you will:

- Understand what a bounded sequence is,
- Know how to tell if a sequence is bounded.

Look at the following sequence,

$$a_n = \begin{cases} 1 + \frac{1}{2^n}, & \text{n odd;} \\ 2^n, & \text{n even.} \end{cases}$$

The first few terms of the sequence are

$$\begin{aligned} a_1 &= 1 + \frac{1}{2} = \frac{3}{2} && (n = 1, \text{odd}) \\ a_2 &= 2^2 = 4 && (n = 2, \text{even}) \\ a_3 &= 1 + \frac{1}{2^3} = 1 + \frac{1}{8} = \frac{9}{8} && (n = 3, \text{odd}) \\ a_4 &= 2^4 = 16 && (n = 4, \text{even}) \text{ ... etc.} \end{aligned}$$

Sequence is $\frac{3}{2}$, 4, $\frac{9}{8}$, 16, ...

It is not so obvious what the next term is. This is why we need the function above to be defined precisely.

The last example is an example of a sequence which is called *unbounded*, i.e. the n^{th} term in the sequence grows without bound.

Definition: A sequence $a_1, \ a_2, \ a_3, ...$ is *bounded* if there exist a number $M > 0$ such that

$$|a_n| < M$$

for every natural number n. This means that regardless of what term we are looking at, the absolute value of that term must be less than M.

In the previous example it is impossible to find a number M that is larger than 2^n for all n and so this sequence is unbounded. Regardless of the value of M that you choose it is still possible to find a value of n such that $2^n > M$.

Look at the following example,

$$\left\{2^{-n}\right\}_{n=1}^{\infty}$$

is an example of a bounded sequence. It is bounded above by 1 - technically any value greater than $\frac{1}{2}$ will bound this sequence. List the first few terms to convince yourself that it is in fact bounded above by 1.

Related Reading

Gersting, J.L. 2007. *Mathematical Structures for Computer Science.* 6^{th} Edition. Freeman & Company.

Morris, O.D. 1992. *Text & Tests 4.* The Celtic Press.