Integration Techniques

Aim

To introduce different techniques of integration.

Learning Outcomes

At the end of this section you will be able to:

- Understand the process of integration by substitution,
- Understand the process of integration by parts.

Integration by Substitution

When an integrand cannot be evaluated by inspection we require one or more special techniques. The most important of these techniques is the **method of substitution**, the inverse of the "Chain Rule" used in differentiation. When differentiating a composite function such as $y = (3x-4)^5$, the *chain rule* is generally used, i.e. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}.\dfrac{\mathrm{d}u}{\mathrm{d}x}$, where $u = 3x - 4$. Thus,

$$
\begin{aligned}
y = (3x-4)^5 \qquad & \Rightarrow y = u^5 \Rightarrow \frac{dy}{du} = 5u^4 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 3 \\
& \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x} \\
& = 5u^4.3 \\
& = 15u^4 = 15(3x-4)^4.
\end{aligned}
$$

Integration by substitution is very similar to reversing the *chain rule* and is used to change an integrand into a form that is easier to integrate.

<u>Example 1</u>

Find

$$\int (3x+1)^4 \ \mathrm{d}x$$

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$$\begin{aligned}
\text{Let } u &= 3x+1 \\
\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} &= 3 \\
\Rightarrow 3 \ \mathrm{d}x &= \mathrm{d}u \\
\Rightarrow \mathrm{d}x &= \frac{1}{3} \ \mathrm{d}u \\
\text{Then } \int (3x+1)^4 &= \int u^4 . \frac{1}{3} \ \mathrm{d}u \\
&= \int \frac{u^4}{3} \ \mathrm{d}u \\
&= \frac{u^5}{15} + c \\
&= \frac{(3x+1)^5}{15} + c
\end{aligned}$$

<u>Example 2</u>

Find

$$\int e^x \sqrt{1+e^x} \ \mathrm{d}x$$

$$\begin{aligned}
\text{Let } v &= 1 + e^x \\
\text{Then } \mathrm{d}v &= e^x \ \mathrm{d}x. \\
\text{Then } \int e^x \sqrt{1+e^x} \ \mathrm{d}x &= \int v^{1/2} \ \mathrm{d}v = \frac{2}{3} v^{3/2} + c = \frac{2}{3}(1+e^x)^{3/2} + c
\end{aligned}$$

Integration by Parts

We can use the method of substitution to integrate products such as $\int 2x \cos(x^2 + 5)dx$, where one of the factors is related to the derivative of the other. However, if the expression to be integrated is a product of two functions, e.g. $\int x \sin x$, where neither factor is related to the derivative of the other, we use a method of integration called **integration by parts**. This technique is in fact the inverse of the product rule for differentiation.

The product rule states that if u and v are two functions of x, i.e. $u(x)$ and $v(x)$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Integrating both sides with respect to x we get

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\ \mathrm{d}x = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x$$
$$\Rightarrow uv = \int u\ \mathrm{d}v + \int v\ \mathrm{d}u$$
$$\Rightarrow \int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$$

Hence the formula for integration by parts is

$$\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$$

(Note: This formula is given on page 42 of the Mathematics Tables).

The advantage of the formula for *Integration by parts* is that it enables us to express one integral of the form $\int u\ \mathrm{d}v$ in terms of another integral $\int v\ \mathrm{d}u$ which could be easier to integrate. It should be noted that this approach will only work in certain cases.

When applying the formula for integration by parts to integrate a product, let one factor of the integrand be equal to u and the other equal to $\mathrm{d}v$. The successful application of the formula depends on the correct choice of u, since this determines whether the second integral, $\int vdu$, is easier to deal with than the first (i.e. $\int u\ \mathrm{d}v$).

Example 3

Find

$$\int x^4 \ln x\ \mathrm{d}x$$

Let $u = \ln x$ and $\mathrm{d}v = x^4\ \mathrm{d}x$.
(Note: If we let $\mathrm{d}v = \ln x\ \mathrm{d}x$ it would result in the integral $\int \ln x\ \mathrm{d}x$ which is not easy to integrate.)

$$u = lnx \quad \text{and} \quad \mathrm{d}v = x^4\ \mathrm{d}x$$
$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \quad \text{and} \quad \int \mathrm{d}v = \int x^4\ \mathrm{d}x$$
$$\Rightarrow \mathrm{d}u = \frac{\mathrm{d}x}{x} \quad \text{and} \quad v = \frac{x^5}{5}$$
$$\text{Recall} \quad \int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$$
$$\Rightarrow \int x^4 \ln x\ \mathrm{d}x = \ln x\Big(\frac{x^5}{5}\Big) - \int \frac{x^5}{5}.\frac{\mathrm{d}x}{x}$$

$$= \frac{x^5}{5}\ln x - \int \frac{x^4}{5}\,\mathrm{d}x$$
$$= \frac{x^5}{5}\ln x - \frac{x^5}{5.5} + c$$
$$= \frac{x^5}{5}\ln x - \frac{x^5}{25} + c.$$

Important:

For definite integrals, the rule for integration by parts becomes

$$\int_a^b u\ \mathrm{d}v = uv\Big|_a^b - \int_a^b v\ \mathrm{d}u$$

Related Reading

Jacques, I. 1999. *Mathematics for Economics and Business.* 3rd Edition. Prentice Hall.

Morris, O.D., P. Cooke. 1992. *Text & Tests 5.* The Celtic Press.

Stewart, J. 1999. *Calculus.* 4th Edition. Brooks/Cole Publishing Company.