Applications to Optimisation

Aim

To demonstrate an application of differentiation called optimisation.

Learning Outcomes

At the end of this section you will:

- Understand what is meant by optimisation,
- Know how to use differentiation to solve optimisation problems.

Optimisation problems consist of a function, the maximum or minimum value of which is to be found, often subject to **constraints** which limit the domain of the function. There are numerous real life applications of optimisation. A businessperson wants to minimise costs and maximise profits. A traveler wants to minimise transportation time. Fermat's Principle in optics states that light follows the path that takes the least time.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimisation problem by setting up a function that is to be maximised or minimised.

The following section will outline a number of steps that should, in most cases, be followed when attempting to solve an optimisation problem.

Steps in Solving Optimisation Problems

1. **Understand the problem.** The first step is to read the problems carefully until it is understood. What are the unknowns ? What are the given quantities ? What are the given conditions ?

2. **Draw a diagram.** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. **Introduce Notation.** Assign a symbol to the quantity that is to be maximised or minimised (call it Q for now). Also select symbols $(a, b, c, ..., x, y)$ for other unknown quantities and label the diagram with these symbols.

4. Express Q in terms of some of the other symbols from Step 3.

5. If Q has been expressed as a function of more that one variable in Step 4, use the given information to find a relationship (in the form of equations) among these variables. Then use these variable to eliminate all but one of the variables in the expression for Q. Thus, Q will be expressed as a function of *one* variable only. Write the domain of this function.

6. Use the methods previously presented to find the absolute maximum or minimum value of the function.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a river. He needs no fence along the river. What are the dimensions of the fence that has the largest area ?

We are trying to maximise area, based on the constraint that we only have 2400 ft of fencing. Let A be the area of the field, and let x and y be the depth and length of the field in ft. We know that $A = xy$ and that $x + x + y = 2x + y = 2400$ (constraint).

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the information that the total length of the fencing is 2400 ft, thus

$$y = 2400 - 2x.$$

We now have,

$$A = x(2400 - 2x) = 2400x - 2x^2.$$

Note that $x \geq 0$ and $x \leq 1200$, so the function we wish to maximise is

$$A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical points we solve the equation

$$2400 - 4x = 0$$

which give $x = 600$. It is clear that $A''(x) = -4 < 0$ and so the critical point at $x = 600$ is a maximum.

Thus, the rectangular field should be 600 ft deep and 1200 ft long.

Related Reading

Stewart, J. 1999. *Calculus.* 4^{th} Edition. Brooks/Cole Publishing Company.