Functions of Two Variables

Aim

To demonstrate how to differentiate a function of two variables.

Learning Outcomes

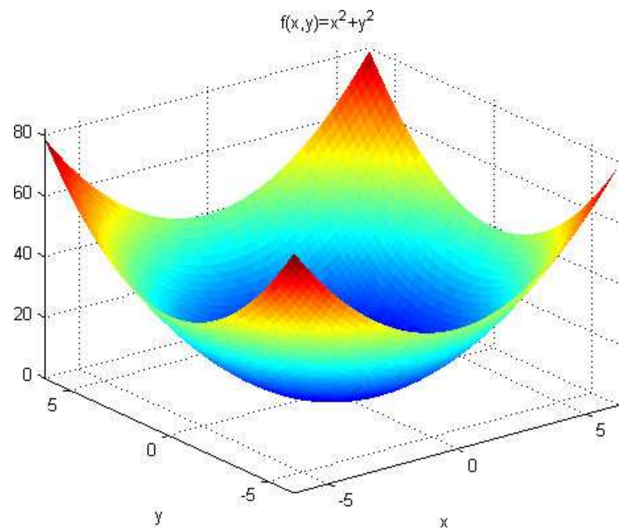
At the end of this section you will:

- Know how to recognise a function of two variables,
- Know how to differentiate functions of two variables.

We have already studied functions of one variable, which we often wrote as $f(x)$. We will now look at functions of two variables, $f(x, y)$. For example,

$$z = f(x, y) = x^2 + y^2.$$

We know that the graph of a function of one variable is a curve. The graph of a function of two variables is represented by a surface as can be seen below. The graph of a function of two variables will always be drawn in three dimensions.



Similar to the definition of a function that we have previously seen, a function of two variables can be defined as a rule that assigns to each incoming *pair* of numbers, (x, y), a uniquely defined outgoing number, z. Therefore, in order to be able to evaluate the function we have to specify the numerical values of both x and y.

Partial Differentiation

Given a function of two variables, $z = f(x, y)$ we can determine two first-order derivatives. The **partial derivative** of f with respect to x is written

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x$$

and is found by differentiating f with respect to x, with y held constant. Similarly the partial derivative of f with respect to y is written

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y$$

and is found by differentiating f with respect to y, with x held constant. We use the partial symbol, ∂, to distinguish partial differentiation of functions of several variables from ordinary differentiation of functions of one variable.

Example 1

Differentiate the following function with respect to x,

$$f(x, y) = x^2 + y^3$$

By the sum rule we know that we can differentiate each part separately and then add the solutions together. When we differentiate x^2 with respect to x we get $2x$. When we differentiate y^3 with respect to x we get 0. To see this, note that y is treated like a constant when differentiating with respect to x. Therefore any function of y, e.g. y^3 is also treated like a constant when differentiating with respect to x and, as we already know, the differential of a constant is 0. Therefore

$$\frac{\partial f}{\partial x} = 2x + 0 = 2x.$$

Example 2

Find both first-derivatives of the following function,

$$f(x, y) = x^2 y$$

Care must be taken in this case because here we have a term consisting of both x and y. To find f_x we differentiate as normal taking x as the variable and y as the constant.

Remember that when we differentiate a constant times a function of x we differentiate the function of x as normal and then multiply it by the constant. For example,

$$3x^2 \quad \text{differentiates to give} \quad 3(2x) = 6x.$$

In our situation, y plays the role of a constant, so

$$x^2y \quad \text{differentiates to give} \quad (2x)y = 2xy.$$

Hence

$$f_x = 2xy.$$

Similarly, to find f_y we treat y as the variable and x as the constant. When we differentiate a constant times y we just get the constant. In our case x^2 plays the role of the constant, so x^2y differentiates to give x^2. Hence,

$$f_y = x^2.$$

It is possible to find second-order derivatives of function of two variables. There are four second-order partial derivatives. The four derivatives are

$$f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}.$$

In general it is true that $f_{xy} \equiv f_{yx}$.

Note: f_{xy} means that we differentiate the function f first with respect to x and then we different the resulting answer, f_x, with respect to y.

Small Increments Formula

To provide an interpretation of a partial derivative let us take one step back for a moment and recall the corresponding situation for functions of one variable of the form

$$y = f(x).$$

The derivative, $\frac{dy}{dx}$, gives the rate of change of y with respect to x. In other words, if x changes by a small amount Δx then the corresponding changes in y satisfies

$$\Delta y \simeq \frac{dy}{dx}\Delta x.$$

The accuracy of the approximation improves as Δx becomes smaller and smaller. Given the way in which a partial derivative is found we can deduce that for a function of two variables

$$z = f(x, y)$$

if x changes by a small amount Δx and y is held fixed then the corresponding change in z satisfies

$$\Delta z \simeq \frac{\partial z}{\partial x}\Delta x.$$

Similarly, if y changes by Δy and x is fixed then z changes by

$$\Delta z \simeq \frac{\partial z}{\partial y}\Delta y.$$

In practice, of course, x and y may both change simultaneously. If this is the case then the net change in z will be the sum of the individual changes brought about by changes in x and y separately, so that

$$\Delta z \simeq \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

This is referred to as the small increments formula or the **total derivative**. If Δx and Δy are allowed to tend to zero then the above formula (which was only an approximation) can be rewritten as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

where the symbols dx, dy and dz are called differentials and represent limiting values of Δx, Δy and Δz, respectively.

Related Reading

Jacques, I. 1999. *Mathematics for Economics and Business.* 3^{rd} Edition. Prentice Hall.