Curve Sketching II

Aim

To demonstrate how to graph a function using differentiation.

Learning Outcomes

At the end of this section you will be able to:

- Find the vertical, horizontal and diagonal asymptotes of a function, if they exist,
- Graph the curve of a function using differentiation.

Asymptotes

Definition: An *asymptote* to a curve is a straight line to which the curve approaches as the distance from the origin increases. It can also be thought of as a tangent to the curve at infinity.

There are three possible types of asymptotes: vertical, horizontal and diagonal(or oblique).

Vertical Asymptotes

Definition: We say that the line $x = a$, where a is a constant, is a vertical asymptote if the function $f(x)$ does not exist at $x = a$, that is

$$\lim_{x \to a^-} = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^+} = \pm\infty$$

When graphing, remember that vertical asymptotes represent x-values that are not allowed. You can never cross a vertical asymptote.

Example 1

Calculate the vertical asymptotes of the following function:

$$f(x) = \frac{1}{x^2 - x}$$

To calculate the vertical asymptotes we look at the denominator of the function and find the values of x where the denominator is zero. The denominator is $x^2 - x = x(x-1)$. Solving this equal to zero gives $x = 0$ and $x = 1$. Therefore there are two vertical asymptotes located at $x = 0$ and $x = 1$.

Horizontal Asymptotes

Definition: We say that the line $y = c$, where c is a constant, is a horizontal asymptote as $x \to \pm\infty$ if

$$\lim_{x \to \pm\infty} f(x) = c$$

Whereas vertical asymptotes can never be crossed, horizontal asymptotes are just useful suggestions. We know that a function will never touch a vertical asymptote whereas with horizontal asymptotes you can (and often do) touch and even cross them. Horizontal asymptotes indicate general behavior far off to the sides of the graph, i.e. they tell you the behaviour of the function as it tends to infinity.

<u>Example 2</u>

Calculate the horizontal asymptote of the following function:

$$f(x) = \frac{1}{x^2 - x}$$

To calculate the horizontal asymptotes we take the limit of the function $f(x)$ as x approaches $\pm\infty$. Therefore

$$\lim_{x \to \pm\infty} \frac{1}{x^2 - x} = \lim_{x \to \pm\infty} \frac{(1/x^2)}{(x^2/x^2) - (x/x^2)} = \lim_{x \to \pm\infty} \frac{(1/x^2)}{1 - (1/x)} = \frac{0}{1} = 0$$

Therefore $y = 0$ is the horizontal asymptote.

Diagonal Asymptotes

Definition: We say that the line $y = mx + c$, where m, c are constants, is a diagonal asymptote as $x \to \pm\infty$ if

$$\lim_{x \to \pm\infty} [f(x) - (mx + c)] = 0$$

To calculate the diagonal asymptote (sometimes called the oblique asymptote) we substitute $y = mx + c$ into the given function and simply. Once this is done equate the function equal to zero and solve for m and c.

Example 3

Calculate the diagonal asymptote of the following function:

$$y = \frac{x^2 - 2x + 2}{x - 1}$$

Substitute $mx + c$ instead of y and simplify;

$$\begin{aligned} mx + c &= \frac{x^2 - 2x + 2}{x - 1} \\ (mx + c)(x - 1) &= x^2 - 2x + 2 \\ mx^2 - mx + cx - c &= x^2 - 2x + 2 \\ x^2(m - 1) + x(-m + c + 2) - c - 2 &= 0 \end{aligned}$$

Equating to zero the coefficients of the two highest powers of x:

$$m - 1 = 0 \rightarrow m = 1$$
$$-m + c + 2 = 0 \rightarrow c = -1$$

Therefore the diagonal asymptote is $y = x - 1$.

Graphing a function

The following procedure should be used when graphing a function using differentiation:

1. Find the critical points,
2. Evaluate the functions f at the critical points,
3. Find the sign of f' between the critical points,
4. Find the roots of f,
5. Find the horizontal/diagonal asymptotes - if any,
6. Find the vertical asymptotes - if any.

Related Reading

Stewart, J. 1999. *Calculus.* 4^{th} Edition. Brooks/Cole Publishing Company.