

# Curve Sketching I

#### Aim

To demonstrate how to graph a function using differentiation.

#### Learning Outcomes

At the end of this section you will be able to:

- Understand the difference between critical points and points of inflection,
- Calculate and classify the critical points and points of inflection of a function.

### **Increasing and Decreasing Functions**

The terms *increasing*, *decreasing* and *constant* are used to describe the behaviour of a function over an interval as we travel left to right along its graph. The function graphed below is said to be increasing on the interval  $(-\infty, 2)$ , decreasing on the interval (2, 4), increasing again on the interval (4, 6) and constant on the interval  $[6, +\infty)$ .



**Definition:** Let f be a function defined on an interval and let  $x_1$  and  $x_2$  be points in that interval.

- f is increasing on the interval if  $f(x_1) < f(x_2)$  whenever  $x_1 < x_2$  for all points  $x_1$  and  $x_2$ .
- f is decreasing on the interval if  $f(x_1) > f(x_2)$  whenever  $x_1 < x_2$  for all points  $x_1$  and  $x_2$ .
- f is constant on the interval if  $f(x_1) = f(x_2)$  for all points  $x_1$  and  $x_2$ .



If tangents were drawn to the graph above you would notice that when f, the function, is increasing its tangent has a positive slope and when f is decreasing its tangent has a negative slope. When f is constant its tangent has zero slope. From this it is possible to arrive at the following result.

**N.B:** Let f be a function that is continuous on an interval [a, b] and differentiable on the open interval (a, b).

- If f'(x) > 0 for every value of x in (a, b), then f is increasing on (a, b).
- If f'(x) < 0 for every value of x in (a, b), then f is decreasing on (a, b).
- If f'(x) = 0 for every value of x in (a, b), then f is constant on (a, b).

#### Concavity

Although the sign of the first derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*. The direction of curvature can be either *concave up* (upward curvature) or *concave down* (downward curvature). The following are two suggested ways to characterise the concavity of a differentiable function f on an open interval:

- f is concave up on an open interval if its tangent lines have *increasing slopes* on that interval and is concave down if they have *decreasing slopes*.
- f is concave up on an open interval if its graph lies above its tangent line on that interval and is concave down if its graph lies below its tangent lines.

Since the slope of the tangent lines to the graph of a differential function f are the values of it's derivative f', the above requirements are the same as saying that f' will be increasing on intervals where f'' is positive and f' will be decreasing on intervals where f'' is negative.

**N.B:** Let f be twice differentiable on an open interval [a, b],

- If f''(x) > 0 for every value x in [a, b], then f is concave up on [a, b],
- If f''(x) < 0 for every value x in [a, b], then f is concave down on [a, b].



## **Inflection Points**

Points where a curve changes from concave up to concave down or visa versa are of special interest. These points are called points of inflection and the following is a more formal definition of what they are.

**Definition:** If f is continuous on an open interval containing a value x and if f changes the direction of its concavity at the point (x, f(x)), then we say that f has an *inflection point at x*.

Note: To find the points of inflection of a function f, simply solve f'' = 0.

### **Relative Maxima and Minima**

Imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima." A relative maximum need not be the highest point in the entire mountain range, and a relative minimum need not be the lowest they are just high and low points *relative* to the nearby terrain.

The relative maxima or minima for all functions occur at points where the graphs of the functions have horizontal tangent lines (slopes equal to zero). A **critical point** of a function f can be defined as a point in the domain of f at which the graph of f has a horizontal tangent line.

Note: To find the critical points of a function f, simply solve f' = 0.

### First Derivative Test

A function f has a relative maximum or minimum at those critical points where f' changes sign.

First Derivative Test: Suppose that f is continuous at the critical point  $x_0$ .

- If f'(x) > 0 on an open interval extending left from  $x_0$  and f'(x) < 0 on an open interval extending right from  $x_0$ , then f has a relative maximum at  $x_0$ .
- If f'(x) < 0 on an open interval extending left from  $x_0$  and f'(x) > 0 on an open interval extending right from  $x_0$ , then f has a relative minimum at  $x_0$ .



• If f'(x) has the same sign on an open interval extending left from  $x_0$  as it does on an open interval extending right from  $x_0$ , then f does not have a relative maximum or minimum at  $x_0$ .

# Second Derivative Test

This is another way (and perhaps an easier way) of classifying critical points that relies on the second derivative of the function f.

Second Derivative Test: Suppose that f is twice differentiable at  $x_0$ .

- If  $f'(x_0) = 0$  and  $f''(x_0) > 0$ , then f has a relative minimum at  $x_0$ .
- If  $f'(x_0) = 0$  and  $f''(x_0) < 0$ , then f has a relative maximum at  $x_0$
- If  $f'(x_0) = 0$  and  $f''(x_0) = 0$ , then the test is inconclusive.

# **Related Reading**

Stroud, K.A. 2001. Engineering Mathematics.  $5^{th}$  Edition. PALGRAVE.

Anton, H., I. Bivens, S. Davis. 2005. Calculus. 8<sup>th</sup> Edition. John Wiley & Sons.