§. INDEFINITE INTEGRAL (ANTIDERIVATIVE)

.1. Definitions

The main task of differential calculus is to find the derivative $f'(x)$ or the differential $df(x)=f'(x)dx$ of the function $f(x)$. The integral calculus solves the inverse problem – finding the function $F(x)$ whose derivative is the given function $f(x)$, $F'(x)=f(x)$ or $dF(x)=F'(x)dx=f(x)dx$.

The integral calculus are applied in Geometry, Mechanics, Physics, Techniques and etc.

Definition. The function $F(x)$, $x\in(a,b)$ is an antiderivative of the function $f(x)$ in the interval (a,b) if it is differentiable $\forall x\in(a,b)$ and $F'(x)=f(x)$ or $dF(x)=f(x)dx$.

Definition. The set of all antiderivative functions of $f(x)$ in a given interval (a,b), $\{F(x)+C\}$, where C is a constant, is indefinite integral of $f(x)$ for all x in (a,b) and it is denoted as

$$\int \boldsymbol{f(x)dx=F(x)+C}.$$

The symbol $\int$ is called ***integral sign***, $\boldsymbol{f(x)}$- ***integrand,*** $\boldsymbol{x}$ - ***variable of integration,*** the symbol $\boldsymbol{dx}$ indicates the variable in which the antiderivative is taken and $\boldsymbol{C}$ - ***constant of integration.***

Rules of Integration.

$$\left(\int f(x)dx\right)'=f(x),$$

$$d\left(\int f(x)dx\right)=f(x)dx,$$

$$\int af(x)dx=a\int f(x)dx,\ a\text{- constant},$$

$$\int\left(f_1(x)\pm f_2(x)\right)dx=\int f_1(x)dx\pm\int f_2(x)dx.$$

$$\int f(x)dx=\frac{1}{A}\int f(x)dAx,\ A\text{- constant},$$

Indefinite Integral (Antiderivative)

$$\int f(x)dx = \int f(x)d(x \pm A),\ A\text{- constant},$$
$$\int f(x)dx = F(x) + C \Rightarrow \int f(u(x))du(x) = F(u(x)) + C,$$

where $u(x)$ is a differentiable function.

<u>General Rules of Integration.</u>

$$d\left(\int f(u)du\right) = f(u)du,$$
$$\int dF(u) = F(u) + C$$
$$\int af(u)du = a\int f(u)du,$$
$$\int (f_1(u) \pm f_2(u))du = \int f_1(u)du \pm \int f_2(u)du,$$

where u is a differentiable function.

<u>Rules for Finding the Antiderivatives.</u>

(1) $$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1,$$

(2) $$\int e^x dx = e^x + C,$$

(3) $$\int a^x dx = \frac{a^x}{\ln a} + C, a \neq 1,$$

(4) $$\int \frac{dx}{x} = \ln|x| + C,$$

(5) $$\int \sin x dx = -\cos x + C,$$

(6) $$\int \cos x dx = \sin x + C,$$

(7) $$\int \frac{d}{\cos^2 x} = tgx + C, x \neq (2k+1)\frac{\pi}{2},$$

(8) $$\int \frac{dx}{\sin^2 x} = -ctgx + C, x \neq k\pi,$$

(9) $$\int tgx\,dx = -ln\left|\cos x\right| + C, x \neq (2k+1)\frac{\pi}{2},$$

(10) $$\int ctgx\,dx = ln\left|\sin x\right| + C, x \neq k\pi,$$

(11) $$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{2a} ln\left|\frac{a+x}{a-x}\right| + C, \ a \neq 0 \\ \frac{1}{a} arctg\frac{x}{a} + C, \quad |x| < a \end{cases},$$

(12) $$\int \frac{dx}{x^2 - a^2} = \begin{cases} \frac{1}{2a} ln\left|\frac{x-a}{x+a}\right| + C, \ a \neq 0 \\ \frac{1}{a} arccotg\frac{x}{a} + C, \quad |x| > a \end{cases},$$

(13) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} arctg\frac{x}{a} + C, \ a \neq 0,$$

(14) $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = ln\left|x + \sqrt{x^2 \pm a^2}\right| + C, \ |x| > |a|,$$

(15) $$\int chx\,dx = shx + C,$$

(16) $$\int shx\,dx = chx + C,$$

(17) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = arcsin\frac{x}{a} + C, \ |x| < |a|,$$

(18) $$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} ln\left|x + \sqrt{x^2 + a^2}\right| + C,$$

(19) $$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} arcsin\frac{x}{a} + C.$$

Maple commands.

```
> int(f,x);
> Int(f,x);
```

where **f** is integrand, x - variable of integration.

Checking of the results from **J:=int(F,x)** is with command:

```
> diff(J,x);
```

.2. Integration by Formulae

There exist many integration formulae. We will use (1)÷(19) and also the rules for integrations and the rules for finding of the antiderivatives.

The integral

$$\int f(x).g'(x)dx$$

is denoted very often as

$$\int f(x).dg(x).$$

This process is carrying out of the function $g'(x)$ under differential.

Example. Evaluate the integral

$$J_1 = \int\left(x^4 + 12x^3 - 3x + 5\right)dx.$$

Mathematical Solution. From formula (1):

$$J_1 = \int x^4dx + 12\int x^3dx - 3\int xdx + 5\int dx =$$

$$= \frac{x^5}{5} + 12.\frac{x^4}{4} - 3.\frac{x^2}{2} + 5 + C = \frac{x^5}{5} + 3x^4 - \frac{3x^2}{2} + 5 + C.$$

Solution with Maple.

```
>J[1]:=int(x^4+12*x^3-3*x+5,x);
```

$$J_1 := \frac{x^5}{5} + 3x^4 - \frac{3x^2}{2} + 5 + C.$$

The results is: $J_1 + C$, i.e. $\frac{x^5}{5} + 3x^4 - \frac{3x^2}{2} + 5 + C$.

It is better to use .

```
>J[1]:=Int(x^4+12*x^3-3*x+5,x)=
 int(x^4+12*x^3-3*x+5,x);
```

$$J_1 := \int \left(x^4 + 12x^3 - 3x + 5\right) dx = \frac{x^5}{5} + 3x^4 - \frac{3x^2}{2} + 5.$$

Checking:

```
>diff(J[1],x);
```

$$x^4 + 12x^3 - 3x + 5.$$

Example. Evaluate the integral

$$J_2 = \int 4 sin^3 x.cos\, xdx$$

Mathematical Solution. From (1) and the rules

$$J_2 = 4\int sin^3 x.\left(cos\, x\right) dx = 4\int sin^3 xd\, sin\, x = 4.\frac{\left(sin\, x\right)^4}{4} =$$

$$= sin^4 x + C.$$

Solution with Maple.

```
>J[2]:=Int(4*sin(x)^3*cos(x),x)=
 int(4*sin(x)^3*cos(x),x);;
```

$$J_2 := \int 4 sin^3 x.cos\, xdx = sin\left(x\right)^4$$

Example. Evaluate the integral

$$I_1 = \int \frac{dx}{\sqrt{1-8x^2}}.$$

Mathematical Solution. From (17) and the rules

$$I_1 = \frac{1}{2\sqrt{2}} \int \frac{dx 2\sqrt{2}}{\sqrt{1-\left(2\sqrt{2}x\right)^2}} = \frac{\sqrt{2}}{4} arcsin\left(2\sqrt{2}x\right) + C$$

Solutions with Maple.

```
>I[1]:=Int(1/sqrt(1-8*x^2),x)=
 int(1/sqrt(1-8*x^2),x);
```

$$I_1 := \int \frac{dx}{\sqrt{1-8x^2}} = \frac{\sqrt{2}}{4} arcsin\left(2\sqrt{2}x\right)$$

Example. Evaluate the integral

$$I_2 = \int \frac{1+cos^2 x}{cos^2 x} dx .$$

Mathematical Solution. From (7) and the rules

$$I_2 = \int \left(\frac{1}{cos^2 x} + 1 \right) dx = \int \frac{1}{cos^2 x} dx + \int 1 dx = tgx + x + C .$$

Solutions with Maple.

```
>I[2]:=int((1+cos(x)^2)/(cos(x)^2),x);
```

$$I_2 := \frac{sin(x)}{cos(x)} + x ,$$

Example. Evaluate the integral

$$I_3 = \int \frac{2x\,sin^2 x + cos^2 x}{sin^2 x} dx ,$$

Mathematical Solution. From (1), (8) and the rules

$$I_3 = \int \left(\frac{2x\,sin^2 x}{sin^2 x} + \frac{cos^2 x}{sin^2 x} \right) dx = 2\int x dx + \int \frac{1-sin^2 x}{sin^2 x} dx =$$

$$= 2\frac{x^2}{2} + \int \frac{1}{sin^2 x} dx - \int 1 dx = x^2 - cotgx - x + C .$$

Solutions with Maple.

```
>I[3]:=int((2*x*sin(x)^2+cos(x)^2)/
sin(x)^2,x);
```

$$I_3 := x^2 - cot\,g(x) - x$$

Example. Evaluate the integral

$$I_4 = \int \frac{dx}{(arcsin\,x)^2 \sqrt{1-x^2}} ,$$

Mathematical Solution. From (1), (17) and the rules

$$I_4 = \int \left(arcsin\, x\right)^{-2} \left(\frac{1}{\sqrt{1-x^2}} \right) dx =$$

$$= \int \left(arcsin\, x\right)^{-2} d\, arcsin\, x = -\frac{1}{arcsin\, x} + C .$$

Solutions with Maple.

```
>I[4]:=int(1/(arcsin(x)^2*sqrt(1-x^2)),x);
```

$$I_4 := -\frac{1}{arcsin(x)}$$

Example. Evaluate the integral

$$I_5 = \int \frac{\sqrt{ln\, x}}{x} dx ,$$

Mathematical Solution. From (4), (1) and the rules

$$I_5 = \int \left(ln\, x\right)^{\frac{1}{2}} \left(\frac{1}{x} \right) dx = \int \left(ln\, x\right)^{\frac{1}{2}} d\, ln\, x = \frac{\left(ln\, x\right)^{\frac{3}{2}}}{\frac{3}{2}} + C =$$

$$= \frac{2\, ln\, x \sqrt{ln\, x}}{3} + C .$$

Solutions with Maple.

```
>I[5]:=int(sqrt(ln(x))/x,x);
```

$$I_5 := \frac{2}{3} ln(x)^{\frac{3}{2}}$$

Example. Evaluate the integral

$$I_6 = \int e^x .sin\, e^x dx .$$

Mathematical Solution. From (5), (1) and the rules

$$I_6 = \int sin\, e^x \left(e^x\right) dx = \int sin\, e^x de^x = -cos\, e^x + C$$

Solutions with Maple.

```
>I[6]:=int(exp(x)*sin(exp(x)),x);
```

$$I_6 := -cos\left(e^x\right)$$

Example. Evaluate the integral

$$I_7 = \int \frac{x^3}{x^8 - 2} dx .$$

Mathematical Solution. From (12) and the rules

$$I_7 = \int \frac{\left(x^3\right)}{x^8 - 2} dx = \frac{1}{4} \int \frac{1}{\left(x^4\right)^2 - \left(\sqrt{2}\right)^2} dx^4 =$$

$$= \frac{1}{8\sqrt{2}} ln \left| \frac{x^4 - 2}{x^4 + 2} \right| + C .$$

Solutions with Maple.

```
>I[7]:=Int(x^3/(x^8-2),x)=
 int(x^3/(x^8-2),x);
```

$$??\, I_7 := \int \frac{x^3}{x^8 - 2} dx \; / \; /$$

.3. Selftraining Problems

Evaluate the integrals:

(1) $$\int \left(3^x + 3^{3x}\right) dx ,$$

(2) $$\int \frac{sin\, x dx}{\sqrt{cos^2 x}} ,$$

Indefinite Integral (Antiderivative)

(3) $\int \frac{e^{2x}}{e^{x}-e^{-x}}dx$,

(4) $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^{2}dx$,

(5) $\int \frac{(arccos\,x)^{3}-1}{\sqrt{1-x^{2}}}dx$,

(6) $\int (5x+1)^{5}dx$,

(7) $\int \frac{dx}{2x-3}$,

(8) $\int \frac{cos\,x}{\sqrt{sin^{3}x}}dx$,

(9) $\int \frac{ln^{3}x}{x}dx$,

(10) $\int \frac{dx}{(x+1)\sqrt{x}}$,

(11) $\int \frac{xdx}{1+x^{4}}$,

(12) $\int \frac{dx}{2\sqrt{x}(4-x)}$,

(13) $\int \frac{sin\,x\,cos\,x}{1+sin^{2}x}dx$,

(14) $\int sin(3-2x)\,dx$,

(15) $\int tgx\,dx$,

(16) $\int \frac{1+ln\,x}{x}dx$,

(17) $\int \frac{x^{3}+x}{x^{4}+1}dx$,

(18) $\int \left(1+x^2\right)^{\frac{1}{2}} dx$,

(19) $\int \frac{2x-\sqrt{arcsin\,x}}{\sqrt{1-x^2}} dx$,

(20) $\int \frac{\sqrt{1-x^2}+\sqrt{1+x^2}}{\sqrt{1-x^4}} dx$.

(21) $\int \frac{1+ln\,x}{2x} dx$.

.4. Selfcontrol Test

(1) $\int \frac{xdx}{1+x^2}$,

(2) $\int e^{cos^2 x} sin\,2xdx$,

(3) $\int \frac{dx}{x^2-4x+13}$,

(4) $\int \frac{dx}{\sqrt{-x^2-2x+8}}$,

(5) $\int \frac{dx}{x.cos^2\left(1+ln\,x\right)}$,

(6) $\int \sqrt[7]{\left(x-7\right)^2} dx$,

(7) $\int cos\,3xdx$.

.5. Questions for selfcontrol

1. Write the definitions of indefinite integral.
2. Write the rules for integration.
3. Write the rules for finding antiderivatives that you know.
4. Explain the meaning of the *Maple* commands: **`int(f,x)`**, **`Int(f,x)`**, **`diff(f,x)`**. Show an example.