§. INTEGRATION BY SUBSTITUTIONS

.1. Integration by Substitution – the main idea

One of the very powerful method for finding integrals analytically is the method of substitution. There are many different substitutions that are very useful for evaluating integrals. Some of the most important substitutions will be presented in the next chapters too.

The main aim of substitution using is to find a new integral equivalent to the given one that is easier to evaluate.

The idea is to change the independent variable x in the integral $\int f(x)dx$ to a new variable t by means of a simple connecting formula $x=\varphi(t)$. If follows from this formula that:

$$(x)'_x=\left[\varphi(t)\right]'_t \Rightarrow 1.dx=\varphi'(t)dt \text{ and } f\left[\varphi(t)\right].$$

Consequently, $\int f(x)dx=\int f\left[\varphi(t)\right].\varphi'(t)dt$ which is hopefully simple to evaluate.

Sometimes is better to use substitution $t=\psi(x)$.

Algorithm for evaluating the integral $\int f(x)dx$ ***by substitution.***

Step 1. Definite the best substitution formula depending on you problem.

Step 2. Substitute t for x in the integrand, calculate dx from the substitution formula and find a new integral $\int f\left[\varphi(t)\right].\varphi'(t)dt$.

Step 3. Evaluate the new integral.

Step 4. Transform the answer $F(t)$ to the variable x.

Maple commands.

```
>I:=int(f,x);
>Int(f,x);
```

evaluating the integral of the function **f** and **x** is a variable;

```
>with(student):changevar(x=t^2,I);
```

substituting the variable **t** for **x** with a formula (in this example $x=t^2$) in the integral **I**.

```
>I1:=value(%);
```

evaluating the last result (integral) that is written in our *Maple* program.

Example. Evaluate the integral

$$I_0 = \int \frac{dx}{2(x+1)\sqrt{x}}.$$

Mathematical Solution.

The substitution is $\sqrt{x} = t > 0 \Rightarrow x = t^2 \Rightarrow dx = 2tdt$. Then

$$I_0 = \int \frac{2tdt}{2(t^2+1)t} = \int \frac{dt}{t^2+1} = arctgt + C = arctg\sqrt{x} + C.$$

Solution with Maple.

```
>I0:=int(1/(2*(x+1)*sqrt(x)),x);
```

$$I0 := arctg\left(\sqrt{x}\right)$$

Detailed Solution with Maple.

1) definite the integral into the program ***STUDENT***:

```
>with(student):
>I0:=Int(1/(2*(x+1)*sqrt(x)),x);
```

$$I0 := \int \frac{1}{2(x+1)\sqrt{x}}dx$$

2) substitute t^2 for x:

```
>changevar(x=t^2,I0);
```

$$\int \frac{2t}{2(t^2+1)\sqrt{t^2}}dt$$

3) evaluate the new integral:

```
>I0:=value(%);
```

$$I_0 = \frac{t.arc\,tan(t)}{\sqrt{t^2}}$$

4) go back to the variable x:

```
>I0:=subs(t=sqrt(x),I0);
```

$$I_0 = arctan\left(\sqrt{x}\right)$$

Example. Evaluate the integral

Integration by Substitutions

$$I_1 = \int \frac{dx}{x\sqrt{x^2-4}}.$$

Mathematical Solution.

The substitution is $\left\{ x = \frac{2}{t} \Rightarrow dx = -\frac{2}{t^2}dt \right\}$. Then

$$\frac{1}{x\sqrt{x^2-4}} = \frac{1}{\frac{2}{t}\sqrt{\frac{4}{t^2}-4}} = \frac{t^2}{4\sqrt{1-t^2}} \Rightarrow$$

$$I_1 = \int \frac{t^2}{4\sqrt{1-t^2}}\left(-\frac{2}{t^2}\right)dt = -\frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}dt =$$

$$= -\frac{1}{2} arcsin t + C = -\frac{1}{2} arcsin\frac{2}{x} + C.$$

Detailed Solution with Maple.

```
>restart: with(student):
>I1:=Int(1/(x*sqrt(x^2-4)),x):
>changevar(x=2/t,I1); I1:=value(%);
>I1:=subs(t=2/x,I1);
```

$$I1 := -\frac{1}{2} arcsin\left(\frac{2}{x}\right).$$

Example. Evaluate the integral

$$I_2 = \int \frac{e^{3x}dx}{\sqrt{1-e^x}}.$$

Mathematical Solution.

The substitution is $\left\{ \sqrt{1-e^x} = t > 0 \right\}$ because to make free the integral function from it.

Then $x = ln\left(1-t^2\right) \Rightarrow dx = \frac{-2t}{1-t^2}dt \Rightarrow$.

$$I_2 = \int \frac{e^{3ln\left(1-t^2\right)}}{t}.\frac{-2t}{1-t^2}dt = -2\int \frac{\left(1-t^2\right)^3}{1-t^2}dt =$$

$$= -2\int\left(1-t^2\right)^2 dt = -2\left(t - 2\frac{t^3}{3} + \frac{t^5}{5}\right) + C =$$

$$= -2\left(\sqrt{1-e^x} - \frac{2}{3}\left(\sqrt{1-e^x}\right)^3 + \frac{1}{5}\left(\sqrt{1-e^x}\right)^5\right) + C.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I2:=Int(exp(3*x)/sqrt(1-exp(x)),x);
>changevar(sqrt(1-exp(x))=t,I2);
>I2:=value(%);
>I2:=subs(t=sqrt(1-exp(x)),I2);
```

$$I2 := -2\sqrt{1-e^x} - \frac{2}{5}\left(1-e^x\right)^{(5/2)} + \frac{4}{3}\left(1-e^x\right)^{(3/2)}.$$

Example. Evaluate the integral

$$I_3 = \int \frac{2\sin x.\cos x}{\cos^4 x - 1} dx.$$

Mathematical Solution.

The substitution is $\left\{\cos^2 x = t\right\}$. Then $dt = 2\cos x.\left(-\sin x\right)dx$.

$$I_3 = -\int \frac{dt}{t^2 - 1} = -arc\,t\,gt + C = arctg\left(\cos^2 x\right) + C$$

Detailed Solution with Maple.

```
>restart:with(student):
>I3:=Int(2*sin(x)*cos(x)/(cos(x)^4-1),x);
>changevar(cos(x)^2=t,I3);
>I3:=value(%);
>I3:=subs(t=cos(x)^2,I3);
```

$$I3 := arctan\,h\left(\cos(x)^2\right).$$

Example. Evaluate the integral

$$I_4 = \int \frac{\sin\sqrt[4]{x}dx}{\sqrt[4]{x^3}}.$$

Mathematical Solution.

The substitution is $\left\{\sqrt[4]{x}=t\right\}$. Then $x=t^4, dx=4t^3dt \Rightarrow$

$$I_4=\int\frac{\sin t.4t^3dt}{t^3}=4\int\sin t dt=-4\cos t+C=$$
$$=-4\cos\sqrt[4]{x}+C.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I4:=Int(sin(x^(1/4))/(x)^(3/4),x);
>changevar(x^(1/4)=t,I4); I4:=value(%);
>I4:=subs(t=x^(1/4),I4);
```

$$I4:=-4\cos\left(x^{(1/4)}\right).$$

.2. Integration of Function $f(x)=\left(ax^2+bx+x\right)$ by Substitution

There will be presented substitutions for evaluating the next kind of integrals:

$$J_1=\int\frac{Mx+N}{ax^2+bx+c}dx,$$
$$J_2=\int\frac{Mx+N}{\sqrt{ax^2+bx+c}}dx.$$

It follows from

$$ax^2+bx+c=\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

that the substitution is $x+\dfrac{b}{2a}=t$ or $x=t-\dfrac{b}{2a}$ and $dx=dt$.

Example. Evaluate the integral

$$I_5=\int\frac{2x-2}{x^2-2x+2}dx.$$

Mathematical Solution.

The substitution is $\{x-1=t\}$. Then $x^2-2x+2=t^2+1$ and

$$I_5=\int\frac{2(t+1)-2}{t^2+1}dt=\int\frac{2t}{t^2+1}dt=\int\frac{1}{t^2+1}d\left(t^2+1\right)=$$

$$=ln\left|t^2+1\right|+C=ln\left|x^2-2x+2\right|+C\,.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I5:=Int((2*x-2)/(x^2-2*x+2),x);
>changevar(x-1=t,I5); I5:=value(%);
>I5:=subs(t=x-1,I5);
```

$$I5:=ln\left((x-1)^2+1\right)$$

```
>simplify(I5);
```

$$I5:=ln\left(x^2-2x+2\right).$$

Example. Evaluate the integral

$$I_6=\int\frac{dx}{x^2+4x+8}\,.$$

Mathematical Solution.

The substitution is $\{x+2=t\}$. Then $x^2+4x+8=t^2+4$ and

$$I_6=\int\frac{dt}{t^2+4}=\frac{1}{2}arctg\frac{t}{2}+C=\frac{1}{2}arctg\left(\frac{x+2}{2}\right)+C\,.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I6:=Int(1/(x^2+4*x+8),x);
>changevar(x+2=t,I6); I6:=value(%);
>I6:=subs(t=x+2,I6);
```

$$I6:=\frac{1}{2}arctan\left(\frac{1}{2}x+1\right).$$

Example. Evaluate the integral

$$I_7 = \int \frac{3x-1}{x^2+4x+5}dx.$$

Mathematical Solution.

The substitution is $\{x+2=t\}$. Then $x^2+4x+5=t^2+1$ and

$$I_7 = \int \frac{3t-7}{t^2+1}dt = 3\int \frac{t}{t^2+1}dt - 7\int \frac{dt}{t^2+1} =$$

$$= \frac{3}{2}\int \frac{1}{t^2+1}d\left(t^2+1\right) - 7arctgt =$$

$$= \frac{3}{2}ln\left|t^2+1\right| - 7arctgt + C =$$

$$= \frac{3}{2}ln\left|x^2+4x+5\right| - 7arctg\left(x+2\right) + C.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I7:=Int((3*x-1)/(x^2+4*x+5),x);
>simplify(changevar(x+2=t,I7));
>I7:=value(%);
>I7:=simplify(subs(t=x+2,I7));
```

$$I7 := \frac{3}{2}ln\left(x^2+4x+5\right) - 7arctg\left(x+2\right).$$

Example. Evaluate the integral

$$I_8 = \int \frac{7-8x}{2x^2-3x+1}dx.$$

Mathematical Solution.

The substitution is $\left\{x-\frac{3}{4}=t\right\}$. Then $2x^2-3x+1=2\left(t^2-\frac{1}{16}\right)$.

$$I_8 = \frac{1}{2}\int \frac{1-8t}{t^2-\frac{1}{16}}dt = -8\int \frac{8t-1}{16t^2-1}dt =$$

$$= 8\int \frac{1}{(4t)^2 - 1} dt - 8.4\int \frac{2t}{(4t)^2 - 1} dt =$$

$$= 2\int \frac{1}{(4t)^2 - 1} d(4t) - 32\int \frac{1}{16t^2 - 1} dt^2 =$$

$$= ln\left|\frac{4t - 1}{4t + 1}\right| - 2\,ln\left|16t^2 - 1\right| + C =$$

$$= ln|4t - 1| - ln|4t + 1| - 2\,ln|4t - 1| - 2\,ln|4t + 1| + C =$$

$$= -ln|4t - 1| - 3ln|4t + 1| + C =$$

$$= -ln|4x - 4| - 3ln|4x - 2| + C =$$

$$= -ln\,4 - ln|x - 1| - 3ln\,2 - 3ln|2x - 1| + C =$$

$$= -5ln\,2 - ln|x - 1| - 3ln|2x - 1| + C = .$$

Detailed Solution with Maple.

```
>restart:with(student):
>I8:=Int((7-8*x)/(2*x^2-3*x+1),x);
>I8:=simplify(changevar(x-3/4=t,I8));
>I8:=value(%);
>I8:=simplify(subs(t=x-3/4,I8));
```

$$I8 := -5ln(2) - 3ln(2x - 1) - ln(x - 1).$$

Example. Evaluate the integral

$$I_9 = \int \frac{2x - 2}{\sqrt{x^2 - 2x + 2}} dx.$$

Mathematical Solution.

The substitution is $\{x - 1 = t\}$. Then $x^2 - 2x + 2 = t^2 + 1$ and

$$I_9 = \int \frac{2(t+1) - 2}{\sqrt{t^2 + 1}} dt = \int \frac{2t}{\sqrt{t^2 + 1}} dt = \int \frac{1}{\sqrt{t^2 + 1}} d\left(t^2 + 1\right) =$$

$$= 2\sqrt{t^2 + 1} + C = 2\sqrt{x^2 - 2x + 2} + C.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I9:=Int((2*x-2)/sqrt(x^2-2*x+2),x);
>I9:=simplify(changevar(x-1=t,I9));
```

```
>I9:=value(%);
>I9:=simplify(subs(t=x-1,I9));
```

$$I9 := 2\sqrt{x^2-2x+2}$$

Example. Evaluate the integral

$$I_{10} = \int \frac{3x-5}{\sqrt{9+6x-3x^2}}dx.$$

Mathematical Solution.

The substitution is $\{x-1=t\}$. Then $9+6x-3x^2 = 3\left(4-t^2\right)$ and

$$I_{10} = \frac{1}{\sqrt{3}}\int \frac{3t-2}{\sqrt{4-t^2}}dt = \frac{3}{\sqrt{3}}\int \frac{t}{\sqrt{4-t^2}}dt - \frac{2}{\sqrt{3}}\int \frac{1}{\sqrt{4-t^2}}dt =$$

$$= -\sqrt{3}\int \frac{1}{2\sqrt{4-t^2}}d\left(4-t^2\right) - \frac{2}{\sqrt{3}}arcsin\frac{t}{2} + C =$$

$$= -\sqrt{3\left(4-t^2\right)} - \frac{2}{\sqrt{3}}arcsin\frac{t}{2} + C =$$

$$= -\sqrt{9+6x-3x^2} - \frac{2}{\sqrt{3}}arcsin\frac{x-1}{2} + C.$$

Detailed Solution with Maple.

```
>restart:with(student):
>I10:=Int((3*x-5)/sqrt(9+6*x-3*x^2),x);
>I10:=simplify(changevar(x-1=t,I10));
>I10:=value(%);
>I10:=simplify(subs(t=x-1,I10));
```

$$I10 := -\frac{2}{3}\sqrt{3}\,arcsin\left(\frac{1}{2}x - \frac{1}{2}\right) - \sqrt{9+6x-3x^2}.$$

.3. Selftraining Problems

Evaluate by substitutions the next integrals:

Integration by Substitutions

$$I_{11} = \int \frac{e^{2x}}{e^{4x}+9}dx\,,$$

Answer. $I_{11} = \frac{1}{6}arctg\left(\frac{e^{2x}}{3}\right)+C$,

$$I_{12} = \int \frac{dx}{\sqrt[3]{x}\left(\sqrt[3]{x^2}-1\right)},$$

Answer. $I_{12} = \frac{3}{2}ln\left|\sqrt[3]{x}-1\right|+C$

$I_{13} = \int \frac{dx}{\sqrt{2ax-x^2}}$, substitution $x = a(1-t)$,

Answer. $I_{13} = \pm arccos\frac{a-x}{a}+C$

$$I_{14} = \int \frac{e^x+1}{e^x-1}dx\,,$$

Answer. $I_{14} = ln\left(e^x-1\right)^2 - x + C$.

$$I_{15} = \int \frac{4x+3}{2x^2+2x+1}dx\,,$$

Answer. $I_{15} = ln\left|x^2+x+\frac{1}{2}\right| + arctg\left(2x+1\right)+C$

$$I_{16} = \int \frac{2x-5}{x^2+4x+5}dx\,,$$

Answer. $I_{16} = ln\left|x^2+4x+5\right| - 9arctg\left(x+2\right)+C$

$$I_{17} = \int \frac{x+1}{3x^2+6x+2}dx\,,$$

Answer. $I_{17} = \frac{1}{6}ln\left|x^2+2x+\frac{2}{3}\right|+C$,

$$I_{18} = \int \frac{x+2}{3x^2+6x+2}dx$$

Answer. $I_{18} = \frac{1}{6} ln\left|x^2 + 2x + \frac{2}{3}\right| + \frac{1}{\sqrt{3}} arctg\left(\sqrt{3}x + \sqrt{3}\right) + C$

$$I_{19} = \int \frac{x-1}{2x-3x^2} dx$$

Answer. $I_{19} = -\frac{1}{3} ln\left|x - \frac{2}{3}\right| + \frac{1}{2} ln\left|\frac{x-\frac{2}{3}}{x}\right| + C$

$$I_{20} = \int \frac{2x-4}{\sqrt{x^2+3x+5}} dx,$$

Answer. $I_{20} = 2\sqrt{x^2+3x+5} - 7\, ln\left|x + \frac{3}{2} + \sqrt{x^2+3x+5}\right| + C$

$$I_{21} = \int \frac{2x+3}{\sqrt{x^2+3x+5}} dx,$$

Answer. $I_{21} = \sqrt{x^2+3x+5} + C$

$$I_{22} = \int \frac{4x+2}{\sqrt{2x^2-x+1}} dx,$$

Answer. $I_{22} = 2\sqrt{2x^2-x+1} + \frac{3}{2} ln\left|\sqrt{2}x - \frac{\sqrt{2}}{4} + \sqrt{2x^2-x+1}\right| + C$

$$I_{23} = \int \frac{dx}{\sqrt{x^2-4x-3}},$$

Answer. $I_{23} = ln\left|x + 2 + \sqrt{x^2-4x-3}\right| + C$

.4. Selfcontrol Test

Evaluate the integrals:

$$I_{24} = \int \frac{cos\sqrt[5]{x}dx}{\sqrt[5]{x^4}},$$

.

$$I_{25}=\int\frac{\sin 4x dx}{\cos^4 2x+4},$$

$$I_{26}=\int\frac{\sin 2x dx}{\sqrt{\cos^4 x+1}},$$

$$I_{27}=\int\frac{e^{\sqrt{2x-3}}}{\sqrt{2x-3}}dx,$$

$$I_{28}=\int\frac{dx}{\sqrt{10x-x^2}},$$

$$I_{29}=\int\frac{3x+3}{2x^2-x-1}dx,$$

$$I_{30}=\int\frac{3x-2}{\sqrt{x^2-2x+3}}dx.$$

.5. Selfcontrol Questions

1) Explain the idea of integration by substitutions.

2) Show an example of integration by substitutions.

3) Explain the meaning of the *Maple* commands:

```
with(student), changevar(x=t^2,I1),
simplify(changevar(x=t^2,I1)),
I11:=value(%),I1:=subs(t=sqrt(x),I1),
I1:=simplify(subs(t=sqrt(x),I1)).
```