1. State and sketch the domain of the following functions.

   (a) $f(x,y) = \frac{1}{x^2+y^2}$
   (b) $f(x,y) = \sqrt{y - 2x^2}$
   (c) $f(x,y) = \sqrt{9 - x^2 - y^2}$
   (d) $f(x,y,z) = \sqrt{4 - x^2 - y^2 - z^2}$
   (e) $f(x,y) = \log_x (x + y + 1)$

2. State the domain and the codomain the following functions.

   (a) $f(x,y) = xy \left(\sqrt{x} + \sqrt{y}\right)$
   (b) $f(x,y) = \sqrt{1 - x^2} + \sqrt{1 - y^2}$
   (c) $f(x,y) = \frac{1}{\sin(x+y)\sin(x-y)}$
   (d) $f(x,y,z) = \cos\left(x^2 + y^2 + z^2\right)$

3. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (in the last case $\frac{\partial f}{\partial z}$, too).

   (a) $f(x,y) = x^2 - 5xy + 3y^2$
   (b) $f(x,y) = e^{\sin x}y$
   (c) $f(x,y) = \frac{xy^2}{x+y}$
   (d) $f(x,y) = \ln(x^2 + y^2)$
   (e) $f(x,y) = x^{2y}$
   (f) $f(x,y) = e^{-x}y$
   (g) $f(x,y) = x \ln y + tg\,(xy)$
   (h) $f(x,y,z) = x^2y - \frac{y}{z} + 2\sqrt[3]{\sin z}$

4. State the second derivatives of the following functions.

   (a) $f(x,y) = ch\,(xy) + x \ln y$
   (b) $f(x,y) = x^3y^5 - 2x^2y^3 + 8x$
   (c) $f(x,y) = e^{-\left(x^2+y^2\right)}$, $P_0(2,0)$

5. Investigate the stationary points of the following functions, deciding in each case whether the point is a maximum or a minimum.

   (a) $f(x,y) = 3x^2 + 2xy + y^2$

(b) $f(x, y) = xy$

(c) $f(x, y) = x^2 - y + e^y$

(d) $f(x, y) = e^{-\left(x^2+y^2-xy\right)}$

(e) $f(x, y) = \frac{1}{x} + \frac{1}{y} + xy$

(f) $f(x, y) = 4x^2 + 2xy - 5y^2 + 2$

6. Estimate maximum and minimum values of the function:

$$f(x, y) = \frac{1}{\sqrt{1 + x^2 + y^2}} - 4x^2 y e^{-\frac{x^2+y^2}{2}}$$

within the square $-3 \leq x; y \leq 3$.

7. Find the maximum and the minimum values, if they exist, of the function:

$$f(x, y) = x^3 + y^3 - 2(x^2 + y^2) + 10$$

on the circle $x^2 + y^2 = 1$.

8. Estimate the maximum and minimum values (if they exist) of

$$f(x, y) = x^2 + 8xy + y^2 - 2$$

subject to the constraint $x + 2y - 1 = 0$.

9. A rectangular tank is open at the top and it is designed to hold $1m^3$ of liquid. Find the sides of the base and the height for which the total area of the bottom and the four sides of the tank is a minimum.

10. A farm erects a fence along three sides of a rectangle in order to make a sheepfold; the fourth side of the rectangular is provided by a hedge already in existence. Find the maximum area of the enclosure thus made if the total length of the fence is to be $80m$.

11. If the area of a closed rectangular box is given, find the dimensions of the box when its volume is a maximum.

12. Find the coordinates of point (points) $P(x, y)$ on the plane (x, y) for which the square sum of distances of it from the points $P_1(1, 3), P_2(5, 4), P_3(-1, 2)$ is a minimum.

13. Sketch the region of the following integrations and change of variables:

(a) $\int\limits_{y=0}^{1}\int\limits_{x=y}^{\sqrt{y}} f(x,y)dxdy$

(b) $\int\limits_{x=1}^{2}\int\limits_{y=x}^{2x} f(x,y)dydx$

(c) $\int\limits_{x=0}^{4}\int\limits_{y=0}^{4x-x^2} f(x,y)dydx$

(d) $\int\limits_{x=-1}^{1}\int\limits_{y=0}^{\sqrt{1-x^2}} f(x,y)dydx$

(e) $\int\limits_{y=0}^{2}\int\limits_{x=y-1}^{y+2} f(x,y)dxdy$

(f) $\int\limits_{x=0}^{1}\int\limits_{y=-x}^{x^2} f(x,y)dydx$

14. Compute the following integrals:

(a) $\int\limits_{y=0}^{\frac{\pi}{2}}\int\limits_{x=1}^{2} x\sin^2 y dxdy$

(b) $\int\limits_{0}^{3}\int\limits_{0}^{3-y} e^{2x+3y}dxdy$

(c) $\iint\limits_{T} \frac{dxdy}{(x+y+1)^2}$, where $T=\{(x,y)\in\mathbb{R}^2, 0\le x\le 1, 0\le y\le 1\}$

(d) $\int\limits_{0}^{1}\int\limits_{y}^{\sqrt{y}} \left(x^2+\sin xy\right)dxdy$

(e) $\int\limits_{0}^{1}\int\limits_{0}^{\pi} \left(\cos\left(y+x\right)+arctg2x\right)dydx$

(f) $\iint\limits_{x^2+y^2\le 9} \left(5-3x+4y\right)dxdy$

(g) $\iint\limits_{x^2+y^2\le 4x} \sqrt{16-x^2-y^2}dxdy$

(h) $\iint\limits_{T} arctg\frac{y}{x}$, where $T=\left\{(x,y)\in\mathbb{R}^2, x^2+y^2\le 9, 0\le y\right\}.$

15. Evaluate the following triple integrals and sketch the regions of the integrations:

(a) $\int\limits_{x=0}^{1}\int\limits_{y=0}^{2}\int\limits_{z=0}^{3} dzdydx$

(b) $\int\limits_{z=-1}^{1}\int\limits_{y=0}^{1}\int\limits_{x=0}^{\sqrt{1-y^2}} (y+xz)\,dxdydz$

(c) $\int\limits_{x=0}^{1}\int\limits_{y=0}^{\sqrt{1-x^2}}\int\limits_{z=0}^{x^2+y^2} xydzdydx$

(d) $\iiint\limits_{V}\frac{dxdydz}{(1+x+y+z)^3}$, where the region V is bounded by the planes: $x+y+z=1$, $x=0$, $y=0$ and $z=0$

(e) $\iiint\limits_{V}\sqrt{x^2+y^2}dxdydz$, where the region V is bounded by the cone $x^2+y^2=z^2$ and by the plane $z=1$

16. Evaluate the following integrals. If it is necessary convert to polar-cylindrical or spherical coordinates.

(a) $\int\limits_{x=0}^{3}\int\limits_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int\limits_{z=-1}^{4} dzdydx$

(b) $\int\limits_{0}^{2}\int\limits_{0}^{\sqrt{2x-x^2}}\int\limits_{0}^{a} z\sqrt{x^2+y^2}dzdydx$

(c) $\int\limits_{-R}^{R}\int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int\limits_{-\sqrt{R^2-x^2-y^2}}^{0} (x^2+y^2)dzdydx$

(d) $\iiint\limits_{V}(x^2+y^2)\,dxdydz$, where the region V is characterized by the inequalities $r^2\le x^2+y^2+z^2\le R^2$ and $0\le z$

(e) $\iiint\limits_{V}\sqrt{x^2+y^2+z^2}dxdydz$, where the region V is bounded by the surface $x^2+y^2+z^2=z$

(f) $\int\limits_{0}^{1}\int\limits_{0}^{\sqrt{1-x^2}}\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dzdydx$

17. Compute the volume of the solids which are bounded by the following surfaces:

(a) $z=x^2+y^2,\qquad z=2x^2+2y^2,\qquad y=x,\qquad y=x^2$

(b) $x^2+y^2+z^2=4,\qquad x^2+y^2=3z$

(c) $z = x + y, \qquad z = xy, \qquad x + y = 1, \qquad x = 0, \qquad y = 0$

(d) $z = 6 - x^2 - y^2, \qquad z = \sqrt{x^2 + y^2}$

(e) $x^2 + y^2 + z^2 = 2az, \qquad x^2 + y^2 \leq z^2$

(f) $\left(x^2 + y^2 + z^2\right)^2 = a^3 x$

(g) $\left(x^2 + y^2 + z^2\right)^3 = 3xyz$

18. Find the divergence and the circulation of the following vector fields:

(a) $\mathbf{v} = (yz, zx, xy)$

(b) $\mathbf{v} = (y \sin z, z \cos x, \cos xz)$

(c) $\mathbf{v} = \mathbf{r} \ln |\mathbf{r}|$, where $\mathbf{r} = (x, y, z)$

(d) $\mathbf{v} = \ln (x + y + z)\, \mathbf{i} + \ln (x + y + z)\, \mathbf{j} + \ln (xyz)\, \mathbf{k}$

(e) $\mathbf{v} = (6x^2 y - 4yz^3, 2x^3 - 4xz^3, -12xyz^2)$

19. Prove the identities:

(a) $curl\,(\mathbf{v} + \mathbf{w}) = curl\,(\mathbf{v}) + curl(\mathbf{w})$

(b) $curl\,(f \cdot \mathbf{v}) = f \cdot curl\,(\mathbf{v}) + (gradf \times \mathbf{w})$

(c) $div\,(f \cdot \mathbf{v}) = f \cdot div\,(\mathbf{v}) + (gradf \cdot \mathbf{w})$

(d) $curl\,(gradf) = \mathbf{0}$

(e) $div\,(curl \mathbf{v}) = 0$