Deep geometrical thoughts from some – until now not published – manuscripts of János Bolyai

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Abstract

This paper presents some genuine geometric studies deciphered from János Bolyai's manuscripts. The most notable is beyond doubt a manuscript which attest that János Bolyai as a young mathematician was dealing not only with the problem of angle trisection, but with the problem of cube duplication, too.

In his concise stile he posed the problem of cube duplication as follows: "Given a cube of side a find a cube of side x such that $x^3 = na^3$." He gave quickly three different solutions. The paper presents these solutions keeping the Bolyai's original notations, along some comments concerning their originality.

1 Introduction

The huge manuscript heritage – more then 10,000 pages – of the famous Hungarian mathematician of XIX-th century János Bolyai is a great source of revelation for the patient and assiduous researcher. In the last couple of years, due to the occasion of the anniversary of 200 years of his birthday (1802), vigorous researches were made by several mathematical historians, including among many others E. Kiss, T. Weszely and the first author of this work. As a result, there were published interesting discoveries which prove that János Bolyai – well known as one of the discoverer of the hyperbolic geometry – was dealing with a significantly broader range of mathematical subjects, including not only geometry but algebra, and number theory as well ([Kis])!

His results are equally deep and beautiful yet unknown, due to his bad fortune to be unable to publish or just disseminate them in the mathematical community of his time. In this respect the quick "rejection" of the brilliant Appendix, received from the great Gauss played a determinative bad role.

The present paper reports an early – this time geometrical – discovery of János Bolyai. This was made by him as student at Vienna.

The importance of what follows it is revealed by the fact that until now it was unanimously asserted by the researchers that János Bolyai was dealing with the angle trisection and the circle squaring problems[1], but NOT with the cube duplication, too. All researchers of János Bolyai's scientific heritage do not mention that he was concerned to this problem, too. Moreover, some authors asserts the contrary! Even the celebrated Paul Stäckel, the first who studied his manuscripts, did not noticed this achievement, despite the fact that he discovered Bolyai's solution to the angle trisection problem – on the same manuscript page we use (see p.6)! The other historians of mathematics only quote Stäckel when they refer to the Bolyai's trisection solution.

In this paper we prove, that this is not the truth. On the contrary, János Bolyai, as a teenager, was dealing successfully with the cube duplication problem, too.

First of all it can be noticed that he considered the problem in the most general form right at the beginning. In his concise stile he posed the problem of cube duplication as follows: "Given a cube of side a, find a cube of side x such that $x^3 = na^3$."

He gave quickly three different solutions. We will present these solutions as well as the pages of manuscripts, keeping in the English translation the Bolyai's original notations. Some comments concerning their possible originality will be stated, as well.

2 Brief Historical Remarks

In the ancient times the Greek mathematicians especially emerged three problems out of the numerous geometric construction problems they were dealing with:

* the cube duplication problem,
* the angle trisection problem, and
* the circle squaring problem.

[1] remember that the solution of the circle squaring problem in the hyperbolic plane is announced right in the title of his famous 26 pages work on hyperbolic geometry, short-titled Appendix

Among this problems the first one had have a special importance in the developing of the space-geometry, from that time on. The problem is related to the plain representation of the perspective, which Greeks used, at least in some stage decor ([Wae]).

What is the statement of the mathematical problem?

Problem: *Construct a cube whose volume is twice that of a given cube!*

More generally in the spirit of the ancient mathematics: Given two numbers (i.e segments) a and b, construct a number (i.e. segment) x such that the ratio of the cubes of the numbers a and x (i.e. the volumes of the cubes) be the given ratio a/b, that is

$$\frac{a^3}{x^3} = \frac{a}{b}.$$

The cube doubling problem is the generic case of the ratio 2. In modern language: find x, the solution of the equation

$$x^3 = 2.$$

The solution of this equation can not be constructed using only straightedge and compass. Instead, using specific curves the problem is solvable, as was shown by several mathematicians even in the ancient age.

The problem of cube duplication has an interesting historical origin, too. A possible issue for its origin is the story that Athenians, got a divine inspiration according to which they should construct an altar of double size of the existing one, in order to get rid of a certain plague. Erathostenes composed a poem and gave a solution ([Wae]), which was immortalized on a stone in a church built by the king Ptolemay.

It is a long series of mathematicians, beginning from the 4th century BC, thus who were dealing with this problem in the ancient time. These are (according [Wae]): Archytas, Eudoxus, Menaechmus, Plato, Erathostenes, Nicomedes, Apollonius, Heron, Philon, Diocles, Sporus, and Pappus. A wonderful overview of the history of the ancient mathematics, including the cube doubling problem can be found in the book of Van der Waerden ([Wae]).

The first step toward the solution of the problem was the discovery of Hippocrates. He reduced the problem to the following form: Find two mean proportional between two given straight line segments (or numbers)a and b, that is find x and y such that

$$a/x = x/y = y/b.$$

As we will see later Bolyai also knew this statement of the problem.

Various solutions to this problem were discovered by the ancient mathematicians. We will give some of them as "profs without words", following the web site ([Fra]), where the reader can find the details of the complete proofs, too.

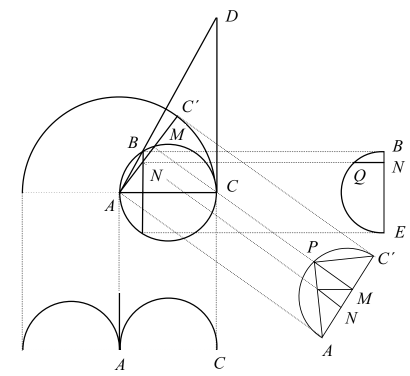

Figure 1: Archytas

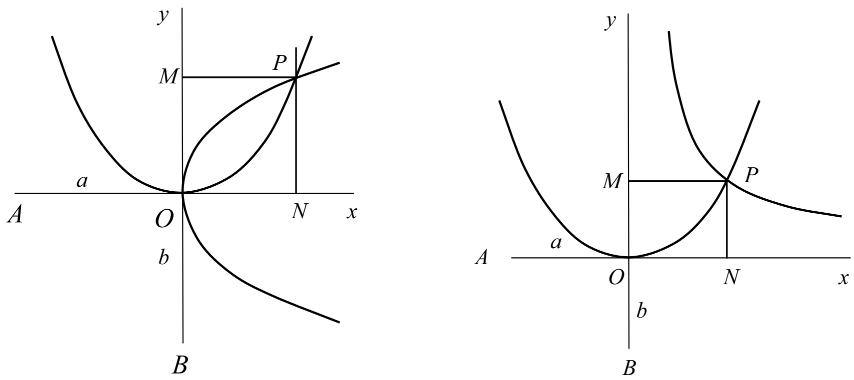

Figure 2: Menaechmus

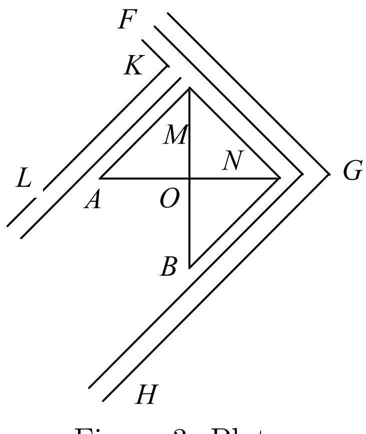

Figure 3: Plato

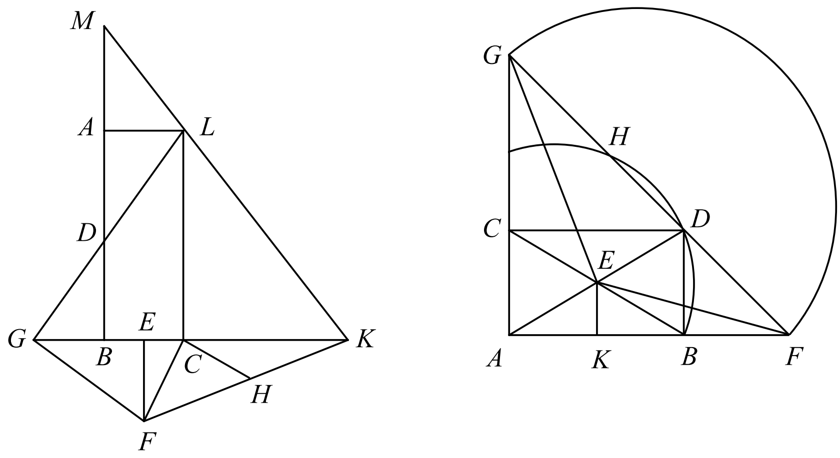

Figure 4: Nicomedes and ... Apollonius, Heron, Philon

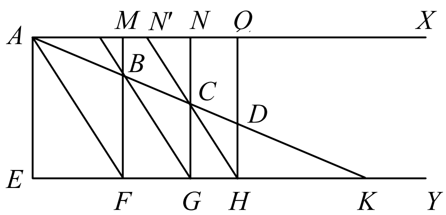

Figure 5: Erathostenes

Finally, let us mention the solution given by Diocles, who discovered the cissoid especially in order to give a solution to the cube duplication problem.

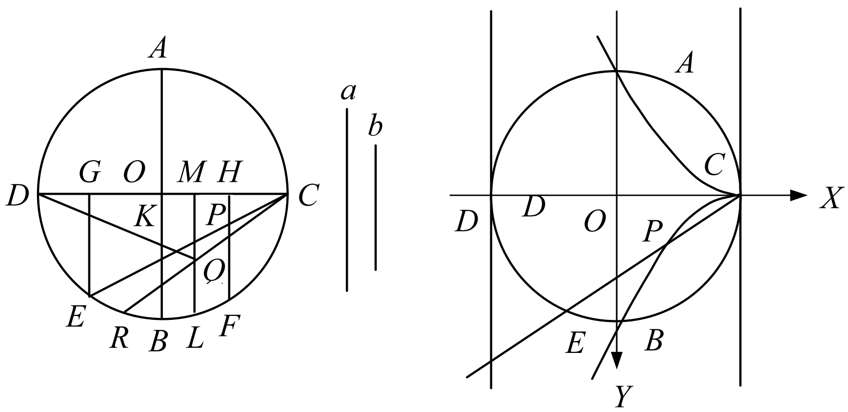

Figure 6: Diocles and ... the cissoid

3 János Bolyai and the cube duplication

Let us begin with the manuscript which proves that János Bolyai (1802-1860) as a student was dealing with the cube duplication problem!

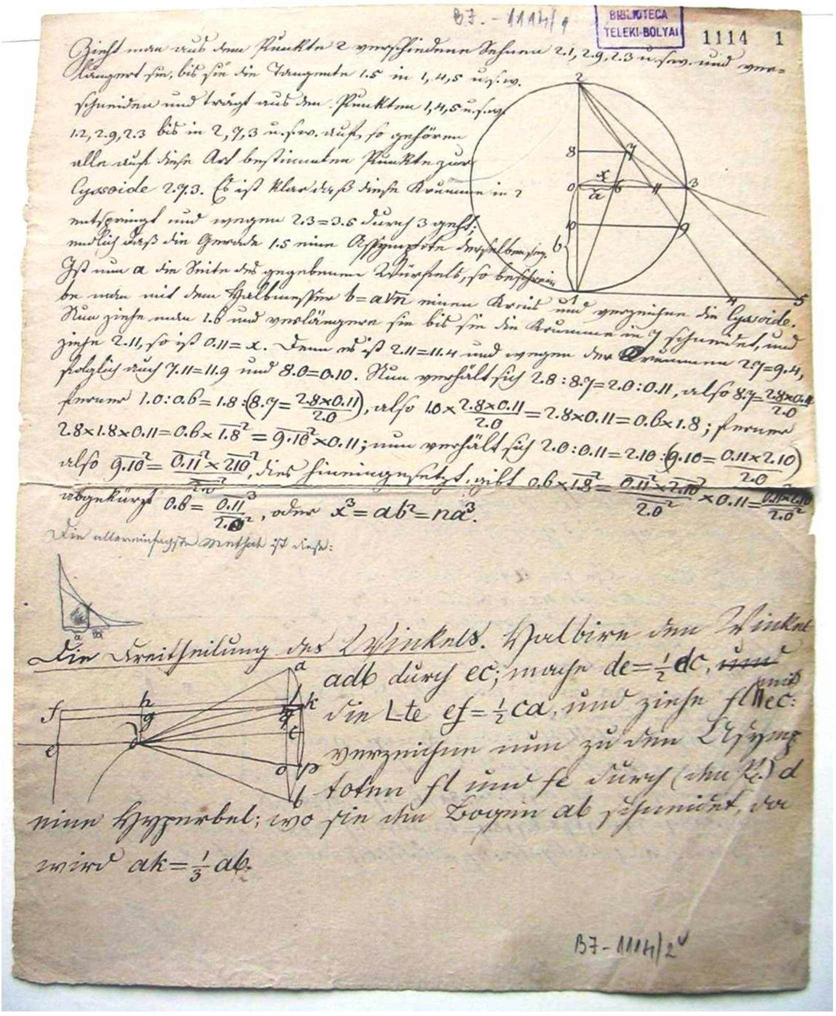

Figure 7: The manuscript, page 1

The first page contains a proof of the cube duplication problem and the angle trisection problem. As the solution is based on the cissoid of Diocles it is obvious that it is inspired from the literature.

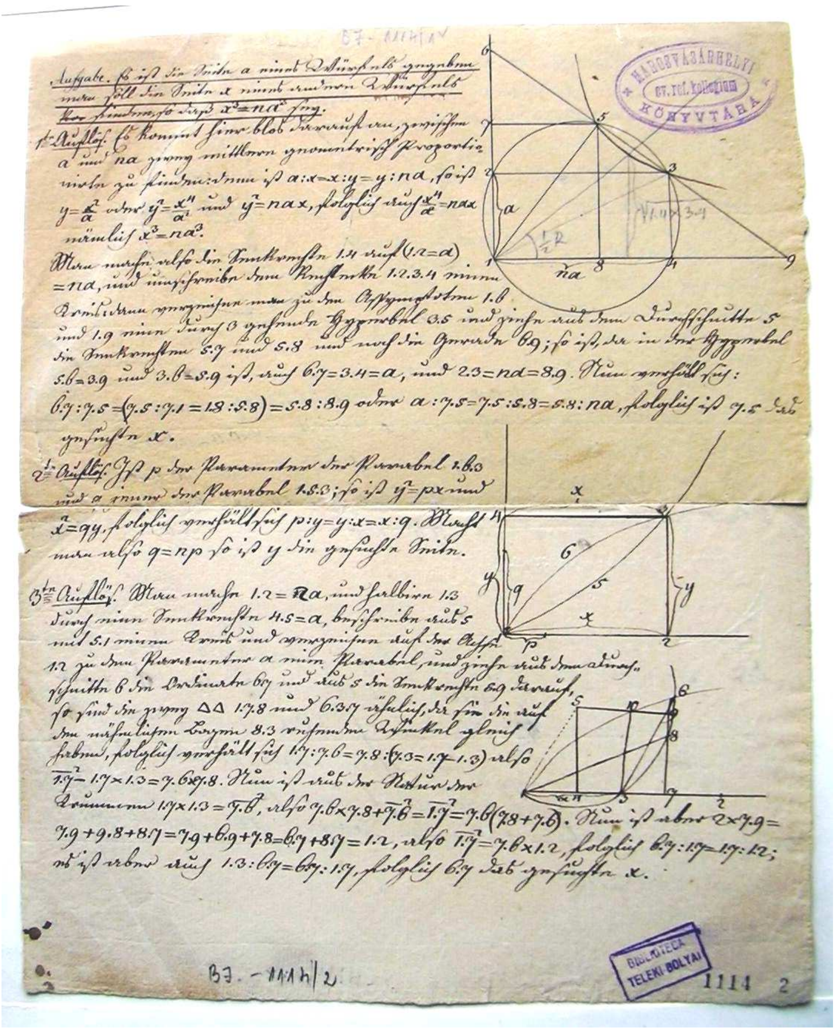

Figure 8: The manuscript, page 2

On the other hand this second page begin with the explicit announcement: I'l give three solutions to the problem. We know that János Bolyai was always very accurate and concerned to give precise references. Later we will refer to this fact.

The language of this manuscripts is German, and it is written using gothic handwritten characters, so it is a difficult job the deciphering. Let us give however the English translation of the whole manuscript.

Here it is the solution based on the use of cissoid.

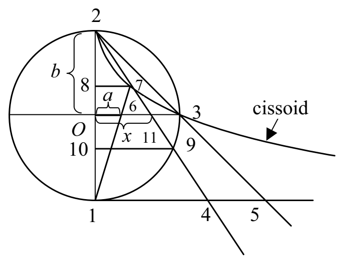

Figure 9: The first drawing on the manuscript page 1

Let us consider a, the length of the edge of a cube and let us draw a circle of radius b=a and center-point 0. We denote the straight line passing across the points 2 and 3 these are János Bolyais notations 2.3 . Let us draw the chord 2.1 2.9 and 2.3 and extend them up to the tangent line 1.5 obtaining the intersection points 1, 4 and 5. This way we can draw the points 2, 7 and 3 on the cissoid. It is clear by the definition of cissoid that 2.3=3.5 and 2.7=9.4, and therefore 7.11=11.9 and 2.8=1.10, 0.8=0.10.

Take the known length a of the cube and place it on the line 0.3 and let a=0.6. Draw the line 1.6 and take its intersection with the cissoid 7. Denote 0.11 by x. Then we have the proportion 2.8:8.7=2.0:0.11 so 8.7=(2.8 x 0.11) / 2.0 and also 1.0:0.6 = 1.8 : (8.7 = (2.8 x 0.11) / 2.0), consequently 1.0 x (2.8 x 0.11) / 2.0 = 2.8 x 0.11 = 0.6 x 1.8; furthermore 2.8 x 1.8 x 0.11 = 0.6 x 1.82 = 9.102 x 0.11; all these lead to 2.0 : 0.11 = 2.10 : (9.10 = (0.11 x 2.10) / 2.0), or .102 = (0.112 x 2.102) / (2.02), from which we can conclude that 0.6 x 1.82 = (0.112 x 2.102) / (2.02) x 0.11 = (0.11 x 2.10) / (2.02), or briefly we just have 0.6 = (0.112) / (2.02), which is x3 = ab2 = na3.

This means that the edge which we were looking for is x, x = 0.11.

Now here are all the three solutions given by János Bolyai to the cube duplication problem.

The first solution:

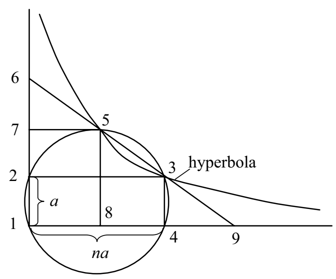

Figure 10: The first drawing on the second manuscript page

Draw the rectangle 1234 so that its height be 1.2 = a and its width be 1.4 = na. Then draw the circle around the rectangle. Take the line 12 as the y axle, and the 14 line as the x axle and then draw the right hyperbola passing through the point 3.

For this is a right angle hyperbola, the asymptotes of the hyperbola are the coordinate axes, namely 1.9 is the x axle and 1.6 is the y axle. Then xy = a.na = 1.4x4.3 = 1.8x8.5. (in general xy=a)

This hyperbola will cut the circle again in the point 5. As the hyperbola is even sided then 5.6=3.9 and 3.6=5.9. From these it follows immediately that 6.7=3.4=a and 2.3=8.9=na. Obviously 1.5 is perpendicular to 6.9, since the angle 215 = angle 158 = angle 532 and therefore 7.52=6.7x7.1 (Theorem of Altitudes in the triangle 156). 7.5:7.1=1.8:5.8 Hence 7.5:7.1=6.7:7.5=1.8:5.8 or 6.7 : 7.5=7.5 : 7.1=1.8 : 5.8=5.8 : 8.9 So we have a : 7.5=7.5 : 5.8=5.8 : 8.9. Therefore the unknown x we searched for is just 7.5.

The second solution:

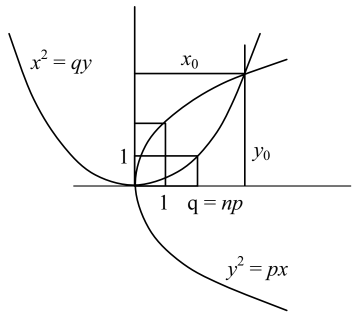

Figure 11: The second drawing on the manuscript page 2

Let p be the parameter of the parabola 1.6.3 and q the parameter of the parabola 1.5.3 therefore y2=px and x2=qy, which imply that p:y=y:x=x:q. Let us put q=np and we get y as the cube side we searched for.

The third solution:

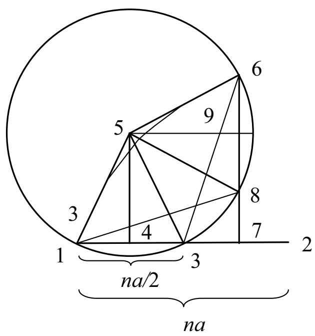

Figure 12: The third drawing on the manuscript page 2

Let us consider 1.2=na. Take the midpoint 3 of the line segment 1.2. The perpendicular bisector of 1.3 is 4.5, such that 4.5=1.3. Draw a circle with center 5 and radius1.5. Take 1.2 as the x axle of a coordinate system and draw the parabola using the length of the line segment 1.3 as the parameter. y2=1.3 x=(na/2)x.

Let us denote the intersection point of the parabola and the circle 6. For the angle 368 = the angle 813, as angles with vertices on the circle, therefore the triangle 178 is proportional with the triangle 637. Therefore 1.7:7.6=7.8:7.3, and on the other side 7.3=1.7-1.3, 1.72-1.7x1.3=7.6x7.8. For the parabola we have 7.62 = 1.3x1.7, then 7.6x7.8+7.62 = 1.72 = 7.6(7.8+7.6). but 2x7.9 = 7.9+9.8+8.7 = 7.9+6.9+7.8 = 6.7+8.7 = 1.2 (we used that 6.9 = 9.8 and 9.7 = 4.5 = 1.3 = na/2) Hence 1.72 = 7.6x1.2, *and it follows 6.7 : 1.7 = 1.7 : 1.2 or 1.3 : 6.7 = 6.7 : 1.7, from here it follows that 6.7 is the edge size x of the cube we were search for!*

Let us make some final remarks. The first and the second solutions are essentially the same as the ancient discoveries, they are probably rediscoveries of the old solutions. Instead, the third solution could be a new one! This opinion is consolidated by the fact that Bolyai started his train of thought with a general n, but he later overwrited n with the digit 2 (you can notice that using a magnifying glass and searching page 2 of manuscript), according to the fact that this solution is a particular one, valid only for $n = 2$, the doubling problem.

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