Mixed product of three vectors

We define the external ternary operation type $V_3 \times V_3 \times V_3 \longrightarrow \mathbb{R}$ in the following way:

Given any three vectors $\overrightarrow{v_1} = \langle a_1, b_1, c_1 \rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2 \rangle$, $\overrightarrow{v_3} = \langle a_3, b_3, c_3 \rangle \in V_3$,their mixed product, (named sometimes triple product) is:

$(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) \longmapsto (\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3}$.

For easier memorization, we can use the determinant:

$$(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Properties

$$(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3} = -(\overrightarrow{v_1} \times \overrightarrow{v_3}) \cdot \overrightarrow{v_2} =$$
$$= (\overrightarrow{v_2} \times \overrightarrow{v_3}) \cdot \overrightarrow{v_1} = -(\overrightarrow{v_2} \times \overrightarrow{v_1}) \cdot \overrightarrow{v_3} =$$
$$= (\overrightarrow{v_3} \times \overrightarrow{v_1}) \cdot \overrightarrow{v_2} = -(\overrightarrow{v_3} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_1}$$

Geometric interpretation

$|(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3}| =$ volume of the parallellipeped spanned by the three vectors, $\overrightarrow{v_1} = \langle a_1, b_1, c_1 \rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2 \rangle$, $\overrightarrow{v_3} = \langle a_3, b_3, c_3 \rangle \in V_3$. The sign of the mixed product expresses if the third vector is in the same "half-space" as the vector product of the first two, i.e. they form a right hand system ($(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3} \geqq 0$) or not.