Scalar product of two vectors

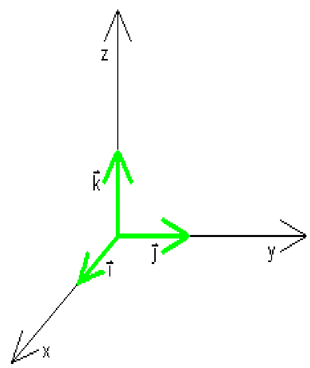
In the sequel we consider only the three dimensional Euclidian vector spaces, denoted by V_3.

Based on the usual notations in $\mathbb{R}^3$, a point P_0 can be written in Cartesian coordinate form as $P_0 = (x_0, y_0, z_0)$.

We will denote by $\overrightarrow{OP_0}$ the oriented line sequence, the position vector of the point P_0, and for which we introduce the similar coordinates $\overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$.

We will have for two different points $P_1 = (x_1, y_1, z_1)$. and P_2 =(x_2, y_2, z_2) in $\mathbb{R}^3$, the oriented line sequence will be denoted by $\overrightarrow{P_1P_2}$, and we will use the coordinate form

$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$, as obviously $\overrightarrow{P_1P_2} = \overrightarrow{OP_2}$ -$\overrightarrow{OP_1}$. The class of conguence $\left\{ \overrightarrow{P_1P_2} = \langle a, b, c \rangle \mid P_1, P_2 \in \mathbb{R}^3 \right\}$ is by definition the vector $\overrightarrow{v} = \langle a, b, c \rangle \in V_3$. We mention 3 special vectors denoted $\overrightarrow{i} = \langle 1, 0, 0 \rangle$, $\overrightarrow{j} = \langle 0, 1, 0 \rangle$, $\overrightarrow{k} = \langle 0, 0, 1 \rangle$, the unit vectors of the 3 coordinate axis, named coordinate vectors.



Basic vector operations

We have $\overrightarrow{v} = \langle a, b, c \rangle = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$.

Scalar product of two vectors (dot product)

We define the external operation type $V_3 \times V_3 \longrightarrow \mathbb{R}$ in the following way:

Given any two vectors $\overrightarrow{v_1} = \langle a_1, b_1, c_1 \rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2 \rangle \in V_3$,their scalar product, (named sometimes dot product) is: $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = a_1a_2 + b_1b_2 + c_1c_2 \in \mathbb{R}$.

Properties

$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = \overrightarrow{v_2} \cdot \overrightarrow{v_1}$ commutativity

$\overrightarrow{v_1} \cdot (\overrightarrow{v_2} + \overrightarrow{v_3}) = \overrightarrow{v_1} \cdot \overrightarrow{v_2} + \overrightarrow{v_1} \cdot \overrightarrow{v_3}$ linearity

$\overrightarrow{v} \cdot \overrightarrow{v} \geq 0$, the last one is used to introduce $|\overrightarrow{v}| = \sqrt{\overrightarrow{v} \cdot \overrightarrow{v}}$, named the length of the vector $\overrightarrow{v}$ (norm).

The unit vector $\overrightarrow{u^0}$ of the vector $\overrightarrow{u}$ is $\overrightarrow{u^0} = \frac{\overrightarrow{u}}{|\overrightarrow{u}|}$, e.g. $\langle 3, 4, 12 \rangle^0 = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$.

The scalar product of two vectors $\overrightarrow{v_1} \cdot \overrightarrow{v_2}$ has an other interpretation:

$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = |\overrightarrow{v_1}|\,|\overrightarrow{v_2}| \cos\varphi$, where φ denotes the angle of the two vectors.

Applications

We deduce: $\cos\varphi = \frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{|\overrightarrow{v_1}||\overrightarrow{v_2}|}$, and we get an equavalent condition for the perpendicularity of two vectors, i.e. the nonzero vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are perpendicular iff $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = 0$.(iff stands here for if and only if).

Vector projection

In order to define the projection of a vector $\overrightarrow{v}$ onto vector $\overrightarrow{u}$ we need first to get the lenght of the projection. If we check the figure below, we observe that $\overrightarrow{v} \cdot \overrightarrow{u^0}$

is exacly what we need, i.e. $\overrightarrow{v} \cdot \overrightarrow{u^0} = |\overrightarrow{v}| \cos\varphi$.

The projection we look for is: $pr_{\overrightarrow{u}} \overrightarrow{v} = \left(\overrightarrow{v} \cdot \overrightarrow{u^0}\right) \overrightarrow{u^0} = \frac{(\overrightarrow{v} \cdot \overrightarrow{u})\overrightarrow{u}}{|\overrightarrow{u}|^2}$.