The Serret-Frénét frame

Let us take the following vector-function (vector valued real function):

$\overrightarrow{r}:\mathbb{R}\longrightarrow V_3$ given by $\overrightarrow{r}(t)=\langle x(t),y(t),z(t)\rangle$, where $x(t),y(t),z(t)$ are the coordinate functions type $\mathbb{R}\longrightarrow\mathbb{R}$. This is the vector which is characterizing the moving point $P\left(x(t),y(t),z(t)\right)$, given with the same coordinate functions in a parametric way. The point $P$ will move along the parametric curve in the space.

It is known from mechanics that the velocity of the point is given by the

$\overrightarrow{v}=\frac{\partial\overrightarrow{r}}{\partial t}=\left\langle \dot{x}(t),\dot{y}(t),\dot{z}(t)\right\rangle$, where $\dot{x}(t)=\frac{\partial x(t)}{\partial t}$ , $\dot{y}(t)=\frac{\partial y(t)}{\partial t}$, and $\dot{z}(t)=\frac{\partial z(t)}{\partial t}$, while its acceleration $\overrightarrow{a}=\frac{\partial^2\overrightarrow{r}}{(\partial t)^2}=\left\langle \ddot{x}(t),\ddot{y}(t),\ddot{z}(t)\right\rangle$, and similarly we have $\ddot{x}(t)=\frac{\partial^2 x(t)}{(\partial t)^2}$ , $\ddot{y}(t)=\frac{\partial^2 y(t)}{(\partial t)^2}$, and $\ddot{z}(t)=\frac{\partial^2 z(t)}{(\partial t)^2}$.

Example: For $\overrightarrow{r}(t)=\langle\cos t,\sin t,e^{-t}\rangle$, we have $\overrightarrow{v}=\langle-\sin t,\cos t,-e^{-t}\rangle$ and $\overrightarrow{a}=\langle-\cos t,-\sin t,e^{-t}\rangle$ .

The Serret-Frénét formulas, which will give us the three vectors called the Serret-Frénét frame can be deduced in the following way:

$\overrightarrow{t}=\frac{\overrightarrow{v}}{|\overrightarrow{v}|}$ (tangent vector),

$\overrightarrow{b}=\frac{\overrightarrow{v}\times\overrightarrow{a}}{|\overrightarrow{v}\times\overrightarrow{a}|}$ (binormal vector), and

$\overrightarrow{n}=\overrightarrow{b}\times\overrightarrow{t}$ (normal vector).

Example: For $t=0$ in the previous curve, we have $\overrightarrow{v}=\langle 0,1,-1\rangle$, and $\overrightarrow{a}=\langle-1,0,1\rangle$, and we compute $\overrightarrow{t}=\left\langle 0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right\rangle$, $\overrightarrow{b}=\left\langle\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$ and $\overrightarrow{n}=\left\langle\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$.