# Knots and Physics in 19th Century Scotland

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Knot theory is now very important in physicists' theory of matter and plays an important role in the development of gauge theories and super-string theory. In a sense we have come full circle, since the theory was first developed in a systematic way as a theory of matter by 19th century Scottish physicists.

## 1. Tait and Maxwell

We start with the interaction between two of these physicists: Tait and Maxwell. It was an interaction which began while they were at school. The two were almost exactly the same age, Tait born on 28 April 1831 in Dalkeith, Scotland, while Maxwell was born on 13 June 1831 in Edinburgh, Scotland. Both attended Edinburgh Academy, entering in the autumn of 1841. However Maxwell enrolled late and had to be put into the year ahead of what would have been the right one for his age. Tait was therefore in the class below Maxwell but the two soon became close friends.

Peter Guthrie Tait

Perhaps the fact that Maxwell was in a class higher affected his performance, or perhaps it was the fact that the shy country boy, nicknamed "Dafty", took a while to adjust to the school that meant that Tait's school career was, at least initially, much more successful than Maxwell's. Tait wrote of Maxwell:

> *At school he was at first regarded as shy and rather dull. he made no friendships and spent his occasional holidays in reading old ballads, drawing curious diagrams and making rude mechanical models. This absorption in such pursuits, totally unintelligible to his schoolfellows, who were then totally ignorant of mathematics, procured him a not very complimentary nickname. About the middle of his school career however he surprised his companions by suddenly becoming one of the most brilliant among them, gaining prizes and sometimes the highest prizes for scholarship, mathematics, and English verse. From this time forward I became very intimate with him, and we discussed together, with schoolboy enthusiasm, numerous schoolboy problems, among which I remember particularly the various plane sections of a ring or tore, and the form of a cylindrical mirror which should show one his own image unperverted.*

James Clerk Maxwell

There is a notebook in which Tait and Maxwell recorded the "schoolboy problems" he referred to in the above quote. Much of the manuscripts are written in beautiful calligraphy. Great care has been taken with these parts and pencil lines have been drawn in and then removed to ensure that all lines are straight and are perfectly left justified. Most of the manuscripts are by Tait and signed '*fecit P G Tait*' with a date. Some, in particular the ones that Tait refers to above, are by Maxwell and signed and dated by him. For example the manuscript on the Conical Pendulum is signed by Maxwell and dated 25 May 47. There are some parts of the notebook written in ordinary handwriting rather than the calligraphy of much of the notebook.


An extract from Tait and Maxwell's schoolboy notebook

The friendship between Tait and Maxwell continued at Edinburgh University and after that, although they saw each other infrequently, they corresponded regularly. The letters are filled with fun and clever jokes showing the great warmth between the two. There is no sign that this warmth was any the less after they competed for the Edinburgh chair. The Chair of Natural Philosophy at the University of Edinburgh became vacant in 1859 when J D Forbes moved to the University of St Andrews. Tait was a candidate for the chair but so was Maxwell. Tait won despite Maxwell's outstanding scientific achievements. When the Edinburgh paper, *The Courant*, reported the result it noted that Tait had been chosen in preference to Maxwell since:

> *... there is another quality which is desirable in a Professor in a University like ours and that is the power of oral exposition proceeding on the supposition of imperfect knowledge or even total ignorance on the part of pupils.*

The claim that Tait was the better person to teach poorly qualified pupils was certainly a fair one and, of course, Tait's personality meant that he made a stronger impression on the appointing committee rather than the much more reserved Maxwell.

**2. Knots**

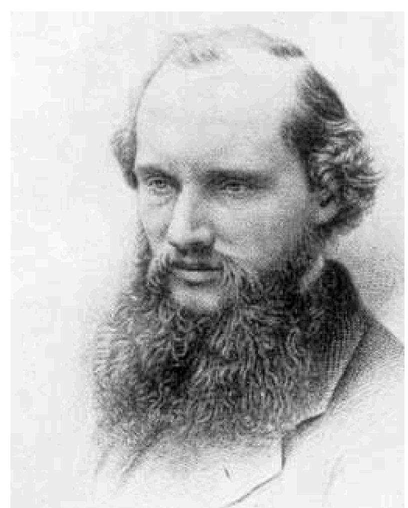
Although van der Monde had written about knots in 1771 and Gauss (according to his note-books) had thought about them as early as 1794 and had looked at the consequences of current flowing in twisted curves, the first significant work on knots was by Listing in his book *Vorstudien zur Topologie* published in 1847. Listing had learnt of topological concepts from his supervisor, Gauss. This work was not seen by either Tait or Maxwell until several years later. The start of Tait's interest was through a paper by Helmholtz. In 1858 Helmholtz published an important paper in *Crelle's Journal* on the motion of a perfect fluid. Helmholtz's paper *Uber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen* began by decomposing the motion of a perfect fluid into translation, rotation and deformation. It was this aspect which first interested Tait who saw that by using Hamilton's quaternions he could express the fluid velocity as a "vector function". Tait had read Hamilton's *Lectures on quaternions* five years earlier. The ideas in Helmholtz's paper which eventually led Tait to study knots concerned vortex lines and vortex tubes. Helmholtz defined vortex lines as lines coinciding with the local direction of the axis of rotation of the fluid, and vortex tubes as bundles of vortex lines through an infinitesimal element of area. Helmholtz showed that the vortex tubes had to close up and also that the particles in a vortex tube at any given instant would remain in the tube indefinitely so no matter how much the tube was distorted it would retain its shape.

Helmholtz was aware of the topological ideas in his paper. He described his theoretical conclusions regarding two circular vortex rings with a common axis of symmetry in the following way:

> *If they both have the same direction of rotation they will proceed in the same sense, and the ring in front will enlarge itself and move slower, while the second one will shrink and move faster, if the velocities of translation are not too different, the second will finally reach the first and pass through it. Then the same game will be repeated with the other ring, so the ring will pass alternately one through the other.*

As we have mentioned, Tait's first interest in Helmholtz's paper was because he saw applications of quaternions there. It was not until 1867 that Tait verified Helmholtz's theoretical claims regarding two circular vortex rings with experiments with smoke rings.

These experiments were to have a major influence on the Scottish physicist and friend of Tait, William Thomson (later Lord Kelvin). Tait and Thomson had cooperated on a major work: *A treatise on Natural Philosophy* in the 1980s. Thomson saw the permanence of form as a possible explanation for atoms and therefore explain the way that the different elements could be built. Although we now know that Thomson was completely wrong with his theory of vortex atoms, it led Tait to consider knots. The vortex atoms, being built from the aether, required no special material. The different elements were accounted for by atoms composed of different knots or links. Oscillations of the knots would, Thomson believed, explain the spectral lines which were characteristic of the different elements.


William Thomson:
Lord Kelvin

**3. Enter Maxwell**

Maxwell had entered the discussions which went on in letters exchanged by the three Scottish mathematical physicists. He was interested in knots because of electromagnetic considerations and in a letter to Tait written on the 4 December 1867 he rediscovered an integral formula counting the linking number of two closed curves which Gauss had discovered, but had not published, in 1833. Maxwell also gave equations in three dimensions which represented knotted curves.

Maxwell stated that he had *amused himself with knotted curves for a day or two*. This couple of days was amazingly fruitful .

He gave the three examples shown at the right

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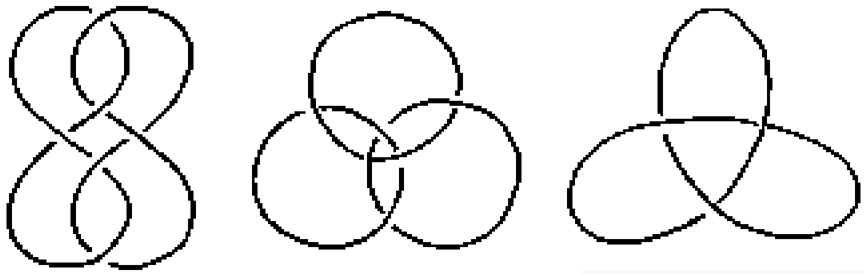
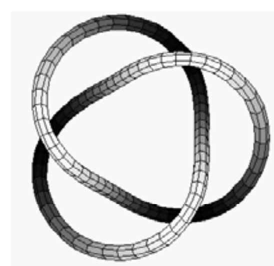
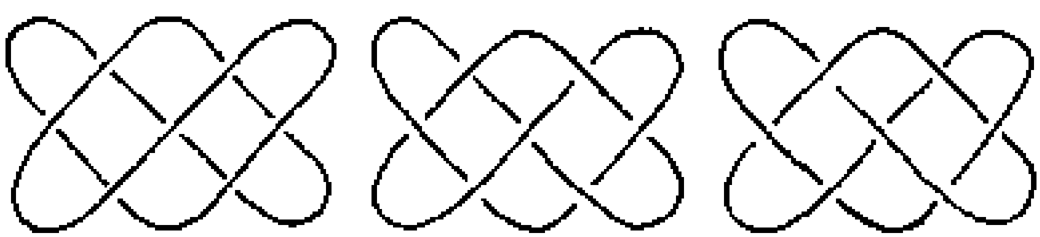


The first of these is a pair of linked curves whose *linking number* (which Maxwell defined using an integral related to his electromagnetic theory) is zero. This is now called the *Whitehead link* after J H C Whitehead who rediscovered it in the 1930's. An equivalent version is shown on the right.

The second example is called the *Borromean rings* (after the Italian family which features them on its mediaeval coat of arms). The configuration is inseparable even though each pair of rings is unlinked.

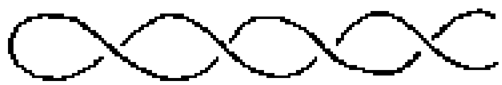
Maxwell gives an equation for the third example: a simple *overhand knot* and remarks that changing the sign of one of the parameters in the equation will change a left-handed into a right-handed example — and notes that these are distinct.
Similar equations can be used to plot the whole family of *torus knots*.

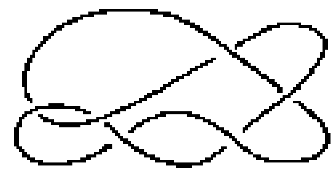




Curves $x = \sin 2p$, $y = \cos 3p$, $z = \sin(5p + \gamma)$ for $0 \le p \le 2\pi$ and various values of $\gamma$.

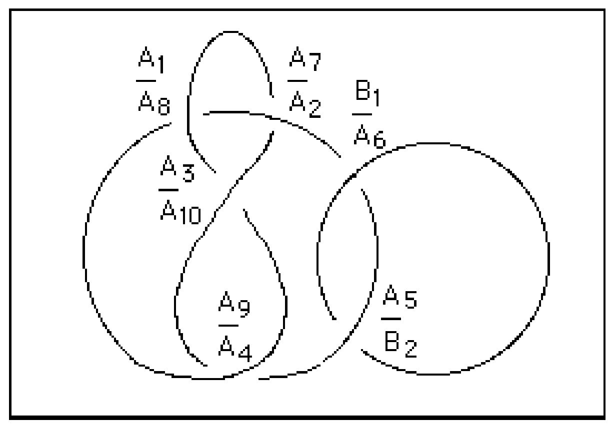
Maxwell then shows how starting from the equation of a Lissajous figure one could construct the set of closed curves shown above. The first ($\gamma = 0$) is unknotted and the others ($\gamma = \pi/3$ and $7\pi/12$) are in fact the same knot ("*knotted in a different way but to the same degree*") — though that is not trivial to prove.





He also notes that this knot can be made in a different way by giving a doubled rope $4\frac{1}{2}\pi$ twists and then linking the ends together. The mathematician John Conway discovered a similar systematic way of constructing knots in the 1950's.

Thus in a couple of days, Maxwell had anticipated much of what would happen in knot theory over the next 80 years.




Maxwell, in three unpublished manuscripts, considered a theory of knots. In the first he considered two-dimensional projections of links and devised a way of coding the diagrams to indicate which curve was above and which below at crossings on the projections. This is essentially what is now called the *Dowker notation* introduced by Hugh Dowker in 1983.

He then looked at ways of modifying the diagrams without changing the link or knot. Maxwell showed that every knot projection had a region bounded by less than four arcs. For a region bounded by one arc he noted that the region could be eliminated by uncoiling the curve. For regions bounded by two arcs, he noted that there were two cases, one where the arcs could be separated and the region eliminated, the other where this could not be done without making changes in other parts of the diagram. For regions bounded by three arcs Maxwell noted that again there were two cases:

> *In the first case any one curve can be moved past the intersection of the other two without disturbing them. In the second case this cannot be done and the intersection of two curves is a bar to the motion of the third in that direction.*

Although his approach contained no mathematical rigour, still it is interesting to note that at this early stage Maxwell had defined the "Reidemeister moves" which would be shown to be the fundamental moves in modifying knots in the 1920s.

These manuscripts by Maxwell were not published at the time they were written despite Tait asking him to submit his ideas on knot theory to the Royal Society of Edinburgh for publication. However, more than 100 years after they were written, these manuscripts were published. There are three manuscripts on knots and some time between the second, which Maxwell wrote in October 1868, and the third, which he wrote on 29 December 1868, he had read Listing's 1847 paper *Vorstudien zur Topologie* for in the third manuscript he lists Listing's main results. In February 1869 Maxwell presented an account of Listing's topological ideas to the London Mathematical Society.

Before leaving Maxwell's contribution we note his admiration for the work of Tait expressed in a letter of 1871 to Thomson:

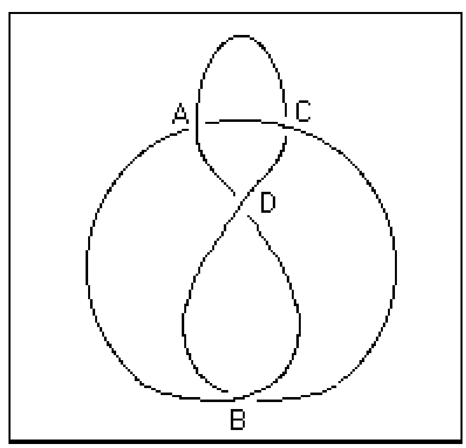
> *You should let the world know that the true source of mathematical methods as applicable to physics is to be found in the Proceedings of the Royal Society of Edinburgh. The volume- surface- and line-integrals of vectors and quaternions and their properties as in the course of being worked out by T′ (Tait) is worth all that is going on in other seats of learning.*

**4. Tait classifies knots**

By 1876 Thomson had made little progress with his ideas of vortex atoms. There were many problems in his way and indeed by this stage he had not succeeded in mathematically describing how two vortex rings would interact if they did not have a common axis of symmetry much more the way that knotted vortices would interact. Also there was no insight into vortex atoms through lists of knots which, in Thomson's theory, would explain the chemical elements. Tait decided to embark on a classification of plane closed curves in 1876, writing in a report to the British Association for the Advancement of Science:

> *The development of this subject promises absolutely endless work - but work of a very interesting and useful kind - because it is intimately connected with the theory of knots, which (especially as applied in Sir W Thomson's Theory of Vortex Atoms) is likely soon to become an important branch of mathematics.*

Now by looking at plane closed curves Tait was considering alternating knots, namely those which when traversing the projection in 2-dimensional space the crossings go alternately over and under. Choosing a starting point and a direction to traverse the path, he labelled the
first, third, fifth etc. points by *A, B, C* etc. A knot with *n* crossings *A, B, C,* ... would then be described by the sequence of crossings of length 2*n* where each of *A, B, C,* ... occurred exactly twice when the knot was

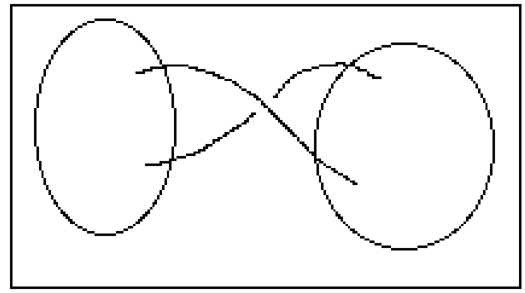



A knot with scheme *ACBDCADB*.
It has reduced scheme *CDAB*
(the letters in the even places).

traversed. Tait called the sequence the "scheme of the knot".

Again this is essentially the Dowker notation.

There were then two basic problems to solve. Firstly which reduced schemes correspond to a knot, and secondly how could it be determined when two knots described by such reduced schemes were "the same". However there were some other problems, for example although reduced scheme of length 5, say, might represent a knot, it might be able to represent it with less than 5 crossings. It might be a knot which could be reduced to one with fewer crossings. For example if the projection contained a crossing which divided the curve into two parts which did not intersect, then this was a nugatory crossing which could be removed by a twist.

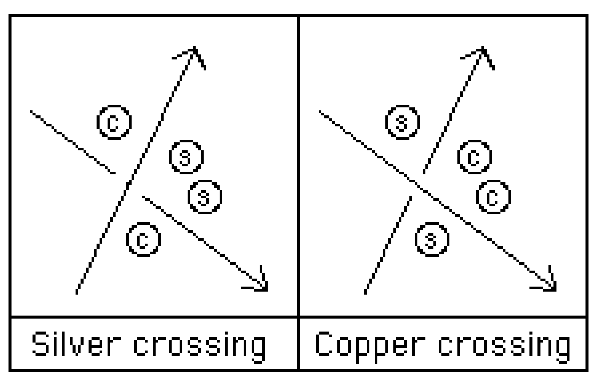


Example of a nugatory crossing.

Tait conjectured that an alternating diagram without nugatory crossings would contain the minimum number of crossings. This became known as Tait's first conjecture. He gave a "proof" which showed that only nugatory crossings allowed the number of crossings to be reduced. However this is not good enough for there might be a sequence of moves which first increase the number of crossings, then further moves reduce to a fewer number of crossings than were there originally. If we interpret Tait in a form that he seems to have used the conjecture, namely that two alternating diagrams without nugatory crossings representing the same prime knot are related by a sequence of twists, then we get what has been called Tait's second conjecture. This was not finally proved until 1993.

Without any rigorous theory, which would have been well beyond nineteenth century mathematics, Tait began to classify knots using his mathematical and geometrical intuition. He knew that what was really required was a knot invariant, that is something which would be independent of the way that the knot was represented in two dimensions. First he looked for numerical invariants and considered the minimal number of crossings that a given knot might have in a two dimensional representation. This would lead him to Tait's first conjecture for alternating knots.

Another idea which seemed promising to Tait was the "beknottedness" which he defined as follows. Travel round the knot diagram and immediately after each crossing throw a copper coin to the left and a silver coin to the right if the crossing was above, or throw a silver coin to the left and a copper coin to the right if the crossing was below.



Tait then defined "beknottedness" (now known as the *twist number*) as the excess of silver crossings over copper ones. If only diagrams without nugatory crossings were considered then Tait believed that this was a knot invariant. In fact it is not, but for alternating knots, it is an invariant and this fact is a consequence of Tait's second conjecture (a theorem since 1993). He tried other more obviously physical ideas such as considering the knot as a circuit and looking at the work done by a magnetic particle carried by a current in the knot. He tried another idea which at first looked very promising to him, namely the minimal number of crossings which required to be changed for under to over (or visa-versa) to unknot the knot. His first thought was that this would be half the beknottedness. He soon saw that this was not so. Seeing that the two concepts were distinct Tait changed his definitions and called the minimal number of crossings which required to be changed to unknot the diagram the beknottedness and he called the minimal number of crossings the knottiness.

By 1877 Tait had classified all knots with seven crossings but he stopped there. He returned to the topic of knots in his address to the Edinburgh Mathematical Society in 1883:

> *We find that it becomes a mere question of skilled labour to draw all the possible knots having any assigned number of crossings. The requisite labour increases with extreme rapidity as the number of crossings is increased. ... I have not been able to find time to carry out this process further than the knots with seven crossings. ... It is greatly desired that someone, with the requisite leisure, should try to extend this list, if possible up to* 11 *...*

**5. Kirkman lends a hand**

Thomas Kirkman read the text of Tait's address and began to work on classifying knots with more than seven crossings. He sent Tait his results on knot projections with up to nine crossings in May 1884 but he had not looked at the problem of deciding which of the projections led to equivalent knots. Tait worked on this side of the problem and, considering only alternating knots, solved the equivalence problems within a few weeks. Tait seemed to know how to tell whether two knots were equivalent without rigorous methods. He states this quite clearly in the paper he wrote tabulating the knots where he says that his methods have:

Rev Thomas Kirkman

> *... the disadvantage of being to a greater or less extent tentative. Not that the rules laid down ... leave any room for mere guessing, but they are too complex to be always completely kept in view. Thus we cannot be absolutely certain that by means of such processes we have obtained all the essentially different forms which the definition we employ comprehends.*

Despite the problems Tait knew exactly what he was doing for, remarkably, his tables are correct. When Kirkman sent him all knot projections with 10 crossings in January 1885 again Tait found all in equivalent knots. The tables were printed in September 1885 and again they are completely correct. By then he had received from Kirkman 1581 knot projections with 11 crossings and this time Tait felt that he did not have the time to solve the equivalence problem for these. However by this time an American mathematician and engineer Charles N Little had sent Tait knot tables which he had calculated and Little began to extend the tables to knots other than alternating ones, and to knots with eleven crossings.

**6. Continuing the story of Knot Theory**

Thomson's vortex atoms were of course abandoned (in favour of the "bee in the cathedral" theory of a nuclear atom) but knot theory continued to fascinate pure mathematicians.

Poincaré introduced the Fundamental group about 1900 and shortly afterwards it was applied to knot complements. Dehn in 1910 introduced generators and relations into group theory to handle the groups that arose in this way. In 1928 Alexander used this algebraic machinery to define the Alexander polynomial — a knot invariant, but not a complete one.

Then not much happened for a long time, until in the 1960s John Conway devised a new method ("tangles") of making and describing knots and extended the Alexander polynomial to links. This led to new computational methods for classifying knots. Indeed, Conway was able to check the calculations which had taken Tait seven years to do in a single afternoon.

In 1984 Vaughan Jones was working on von Neumann algebras and as an offshoot of his theory devised a two-variable generalisation of the Alexander-Conway polynomial. This was immediately generalised by several different groups to the HOMFLY polynomial (named after the initial letters of its inventors).

Because of the way to Jones polynomial arose, it and its generalisations have implications in statistical mechanics and quantum mechanics. In 2004 Michael Atiyah won the first Neils Abel Prize for work he did with Israel Singer on their Index theorem. Atiyah has now retired from being Master of Trinity College Cambridge (Maxwell's college) and is now working in the *James Clerk Maxwell Building* of Edinburgh University. One of his recent works is an influential book *The Geometry and Physics of Knots* describing how the recent advances in knot theory have influenced physicists' attempts to explain their universe. Tait, Maxwell and Thomson would be very pleased to learn that knot theory has now returned to the centre of physical science where they first found it!

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