HOW THE GREEKS MIGHT HAVE DISCOVERED AND APPROXIMATE IRRATIONAL NUMBERS

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Abstract

The lack of original sources opens the door for different, even opposite interpretations of the possible train of thought behind the above results. It is widely accepted that the discovery is due to the early Pythagoreans who tried to find a common measure for the diagonal and side of the square. It turned out that they were incommensurable meaning in modern sense, that 2 is not a rational number. Further, we know from Proclus that the Pythagoreans used proposition II.10 of the Elements to approximate the diagonal-side ratio (2) by rational numbers.

We will show that both the discovery and the approximation can be proved from the same figure using only II.10 unlike the other deductive reconstructions. From our method two alternating approximations can be deduced leading to two lower and upper approximations of 2 by non-periodical decimal fractions. This geometrical approach helps the teacher to form a descriptive idea on the abstract concept of irrational number, and especially on Cantor-axiom.

History of the problem

The emergence of the above problems go back to pre-Platonic times, namely to the Pythagoreans, who "were the first to take up mathematics" according to Aristotle [Met 985b]. The problem is that we have no original sources from this period, only some fragments quoted by later commentators, especially from Proclus. Besides, the works of Plato and Aristotle contain some mathematical allusions, but they are sometimes obscure, and even have different translations.

In Pythagorean mathematics the reciprocal or successive subtraction method (antanairesis) seems to play an important role. This method served for finding the greatest common divisor of two "numbers" meaning always positive integers for the Greeks, and represented by pebble rows in earlier times. Today the method is known as Euclidean algorithm for finding the greatest common divisor of two numbers. The algorithm for numbers is clearly finite. The Greeks called its result the common measure of the original two numbers. By this method every two numbers proved to be commensurable. The name "common measure" shows that antanairesis was connected later to measurement and geometry, too. To support our view let us quote the words of Proclus and Diogenes Laertius:

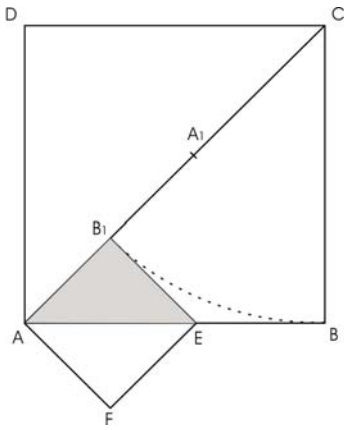
*"The theory of commensurable magnitudes is developed primarily by arithmetic and then by geometry in imitation of it. This is why both sciences define commensurable magnitudes as those which have to one another the ratio of a number has to a number, and this implies that commensurability exists primarily in numbers." [**5**, p. 49]*

*"The Pythagoreans were the first to make inquiry into commensurability, having first discovered it as a result of their observation of numbers; for though the unit is a common measure of all numbers, they could not find a common measure of all magnitudes." [**6**, p. 215]*

Thus we can conclude that the Pythagoreans wanted to find the common measure of the diagonal (b) and side (a) of a square by their antanairesis method. Further, they were convinced that some common measure had to exist according their number based philosophy: everything has a number; every ratio can be expressed as the ratio of two numbers. When some of them (Hippasus?) discovered that the antanairesis is not finite but cyclical, it caused a shock for them. There are different reconstructions in literature of this discovery.

Usual deductive reconstructions of discovery

To show that the antanairesis is cyclical, Figure1 is normally used in literature. The similarity of the triangles AEB_1 and ABC based on the equality of their triangle shows that the procedure is endless. Alternatively, since the triangles CB_1E and CEB are congruent, so $B_1E=EB$, consequently $AE=A_1C$ which proves also that the antanairesis is cyclical.
These proofs are trivial for us but not for the early Pythagoreans who probably have not a proper similarity concept namely an exact criteria for equal ratio. In the Elements the similarity of rectilineal figures is defined only in Book VI based on the concept of "proportionality" formulated in definition 5 of Book V. Further, the criterion by which the congruency of the two triangles in question can be verified was not known for them. In proposition IV.12, even Euclid used Pythagorean Theorem to show the equality of two lines as B_1E and EB, instead of the criterion



Steps	Longer	Shorter	Difference
1	$AC=b_1$	$AB=CB=CB_1=a_1$	$AC-B_1C=AB_1=a$
2	B_1C	$A B_1=A_1 B_1$	$B_1C-A_1 B_1=A_1C$
3	$A_1C=AE=b$	$A_1 B_1=AB_1$	*Step 1 again*

Figure 1. Antanairesis for the square

Remark that the cyclic antanairesis did not convince the Pythagoreans on the incommensurability of the diagonal and side of the square. As philosophers they were atomists so denied the infinite divisibility of the line, generally of continuous quantities. The belief of the existence of a smallest measure was common that time. "For it must seem to everyone a matter for wonder that there should exist a thing which is not measured by the smallest possible measure?"- writesAristotle in Metaphysics 983a, see [**1**]. We claim that the first generally accepted proof must have been the indirect one, Aristotle refers to in his Prior Analytics 41A23-30 [**1**]: "the diagonal of the square is incommensurable with the side, because odd numbers are equal to evens if it is supposed to be commensurate." This logical proof was included in some edition of the Elements, and does not use antanairesis or any geometrical tool except the Pythagorean Theorem. This proof "bypassed" the problem of infinite divisibility.

One can find the train of thought of this proof in many high school textbooks, but without any geometrical tools. The original Greek proof seems to be more didactical. Suppose that b and a commensurable, and denote by e their greatest common measure found by antanairesis. Then b and a are some multiple of e, say b=ne, a=me. Here (n, m)=1, so at least one of them must be odd. By the Pythagorean Theorem the square on b is double of the square on a, i.e. $b^2=2a^2$, consequently $n^2=2m^2$. There follows from this equality that n is even (say n=2k), hence m is odd. Substituting n=2k in the equality, then simplifying by 2 we get 2k=m meaning that m is even, too, which is impossible.

Discovery of incommensurability by II.10

It is generally accepted that the theorems of Book II are very olds. Some say they are Babylonian algebraic identities in geometrical forms, which were well known for the Pythagoreans. We think that the Pythagoreans used to prove that the antanairesis is cyclic for the diagonal and side of the square; moreover, they discovered it in trying to prove it (see Figure 2).

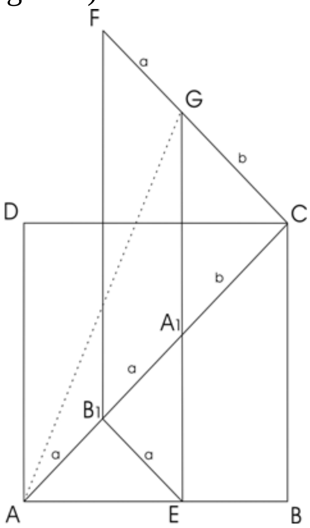


Figure 2. Prop. II.10

One can guess from the figure that A_1C and B_1A_1 again the diagonal (diameter) and side of a square again, so Step 3 repeats Step 1 meaning that the antanairesis is not finite. To verify this conjecture we have to prove that $AC^2=2CB_1^2$ implies $A_1C^2=2A_1B_1^2$; or shortly that $b_1^2=2a_1^2$ implies $b^2=2a^2$. To use up the condition, construct the right triangle CFB_1. The triangles B_1CF and ABC are congruent by proposition I.4, so $FB_1=AC$. Draw a line parallel to B_1F through A_1. Then $FB_1=GE$ easily follows form the simple properties of parallelograms found in Book I. Now, apply the Pythagorean Theorem for AG in the triangles ACG and AEG:

$$AG^2=AE^2+EG^2=2a^2+2a_1^2;\ AG^2=AC^2+CG^2=b_1^2+b^2,$$

so

$$\mathbf{2a^2+2a_1^2=b_1^2+b^2},\ \text{where}\ \mathbf{a_1=b+a,\ b_1=2a+b},$$

which is II.10 in algebraic form. By II.10, $b_1^2=2a_1^2$ implies $b^2=2a^2$, which proves our conjecture.

Euclid formulated this theorem geometrically: “If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.” His proof is identical with ours except he used only the polygon AEA_1CG from our figure. Euclid’s method was deductive, so he did not say anything the inductive background of this theorem. Further, he nowhere used II.10 in the Elements, so its inclusion seems useless.

Approximation of □2 by II.10

Let us quote two ancient sources to illustrate that the Pythagoreans kept trying to express the b/a ratio by numbers even after they proved to be incommensurable. Theon Smyrna writes on their motivation and approximating numbers as follows [**6**]:

“Even as numbers are invested with power to make triangles, squares, pentagons and the other figures, so also we find side and diameter ratios appearing in numbers in accordance with the generative principles; for it is these which give harmony to the figures. Therefore since the unit, according to the supreme generative principle, is the starting point of all the figures, so also in the unit will be found the ratio of the diameter to the side. To make this clear, let two units be taken, of which we set one to be a diameter and the other a side, since the unit, as the beginning of all things, must have it in its capacity to be both side

and diameter. Now let there be added to the side a diameter and to the diameter two sides, for as often as the square on the diameter is taken once, so often is the square on the side taken twice. The diameter will therefore become the greater and the side will become the less."

The second source, Proclus, besides confirming Theon's words, speaks about the role of II.10 in this matter, as well as on the accuracy of the approximation [**5**]:

"…in the squares whose sides they are, [the square of the diagonal] is either less by a unit or more by a unit than double ratio which the diagonal ought to make: more, as for instance is 9 than 4, less as is for instance 49 than 25. The Pythagoreans put forward the following kind of elegant theorem of this, about the diagonals and sides, that when the diagonal receives the side of which it is diagonal it becomes a side, while the side, added to itself and receiving in addition its own diagonal, becomes a diagonal. And this is demonstrated by lines (grammikōs) through the things in the second [book] of Elements by him "

Really, if we start from a smaller square where $b^2=a^2$, then $b_1^2=2a_1^2$ follows for the bigger square with $b_1=2a+b$ and $a_1=b+a$, from II.10. Proclus' words contains the relations $a_1=b+a$, $b_1=2a+b$ shown in Figure 2, that lead to the following recursion formulas

$$\mathbf{a_{n+1}=a_n+b_n,\ b_{n+1}=2a_n+b_n\ ,\ n \geq 0,\ a_0=a,\ b_0=b.}$$

Theon of Smyrna took a=b=1 to start the approximation of the ratio b/a by b_n/a_n. Another possibility is to start with a=1, b=2, since the length $\sqrt{2}$ of b is between 1 and 2. The following table shows the corresponding values for a_n, b_n in both cases, as well as their squares and the approximating values b_n/a_n:

n	a_n	b_n	a_n^2	b_n^2	b_n/a_n	a_n	b_n	a_n^2	b_n^2	b_n/a_n
0	1	1	1	1	1/1	1	2	1	4	2/1
1	2	3	4	9	3/2	3	4	9	16	4/3
2	5	7	25	49	7/5	***7***	***10***	49	100	10/7
3	12	17	144	289	17/12	17	24	289	576	24/17
4	29	41	841	1681	41/29	41	58	1681	3364	58/41
5	70	99	4900	9801	99/70	99	140	9801	19600	140/99

The approximation values 3/2, 7/5, 17/12 from Case 1 were known in ancient Greek, and can be found in some sources: Plato, Theon, Proclus, etc. The situation is different with Case 2 values: there are indirect evidences for their usage. One can find indirect allusions of the use of 10/7 in Codex Constantinopolitanus. Further, Health wrote in [**4**]: "Heron takes 10 as an approximation of 7 or $\sqrt{98}$ ". This clearly shows the use of 10/7 by Heron. Remark, that the Case 1 values are called side and diagonal numbers, too. For more details, see [**3**].

The relation $2a_n^2-b_n^2=(-1)^{n+1}$ can be guessed for Case 1. Proclus also referred to this connection in the quoted section. By altering this relationship we can show that Case 1 values really approximate alternatively 2:

$$2_n^2-b_n^2=(-1)^{n+1} \qquad \frac{b_n^2}{a_n^2}-2=\frac{(-1)^{n+1}}{a_n^2}\to 0,\ \text{as n} \quad .$$

Thus 2 can be understood as the single seed of the following series of nested intervals according to Cantor's axiom:

[1; 3/2=*1,5*], [7/5 =*1,4*; 17/12=*1,416*], [41/29=*1,413*; 99/70=*1,414*],…..

Similarly, Case 2 leads to another series of nested intervals determining also 2

[**4/3**=*1,3*; **2**], [**24/17**=*1,*411;**10/7**=*1,428*], [**140/99**=1,4141; **58/41**=*1,4146*,],….

References

1. Aristotle: Metaphysics, Prior Analytics. In: Great Books of the Western World, Vol. 8, Chicago, Encyclopedia Britannica, 1982.
2. Euclid: Elements, tr. T.L. Heath, in: Britannia Great Books, Vol. 2, Chicago, Encyclopedia Britannica, 1982
3. Filep, L: Pythagorean side and diagonal numbers. AMAPN, 15(1999), 1-7.
4. Heath, T: A history of Greek mathematics, 2 vols, Dover, 1981.
5. Proclus: A Commentary on the First Book of Euclid's Elements. Tr. G.N. Morrow, Princeton University Press, 1970
6. Thomas, I. (tr. & ed.): Selections illustrating the History of Greek Mathematics, Vol. 1. Harvard University Press, 1957