1 Limit and continuity of multivariable functions

Let $X(x_1, x_2, \ldots, x_n)$ and $Y(y_1, y_2, \ldots, y_n)$ be two points of $\mathbb{R}^n$. The *distance* of X and Y is the nonnegative number

$$d(X,Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots (x_n - y_n)^2}.$$

(We can denote this distance by $|X - Y|$, too.)

The set $B(X,r) = \{A \in \mathbb{R}^n : |A - X| < r\}$ is called the *open ball of radius* r *centered at* X. The *closed ball of radius* r *centered at* X is the set $\overline{B}(X,r) = \{A \in \mathbb{R}^n : |A - X| \leq r\}$.

Now suppose $T \subset \mathbb{R}^n$. A point $P \in T$ is called an *interior point of* T if there is an open ball $B(P,r) \subset T$. The set of all interior points of T is called the *interior of* T and is denoted by *int*T.

In the following we consider the function of two variables.

Definition 1. Let $f : T \to \mathbb{R}$ be a function, such that $T \subset \mathbb{R}^2$and suppose $P_0(x_0, y_0) \in \mathbb{R}^2$ is such that that every open ball centered at $P_0(x_0, y_0)$ meets the domain T. If $A \in \mathbb{R}$ is such that for every $\varepsilon > 0$ there exists a real number $\delta > 0$ so that

$$|f(x,y) - A| < \varepsilon$$

whenever

$$0 < (x - x_0)^2 + (y - y_0)^2 < \delta^2,$$

then we say that A is the *limit of* f *at* P_0.

This limit is written

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A \qquad \text{or} \qquad \lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = A.$$

The usual properties of limits are easy to establish, hence the following equalities hold

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) &= \lim_{(x,y)\to(x_0,y_0)} f(x,y) + \lim_{(x,y)\to(x_0,y_0)} g(x,y), \\ \lim_{(x,y)\to(x_0,y_0)} \alpha f(x,y) &= \alpha \lim_{(x,y)\to(x_0,y_0)} f(x,y) \quad (\text{where } \alpha \in \mathbb{R}) \end{aligned}$$

if we suppose that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y)$ exist.

It is important to remark that one may not calculate the limit of a two-variable function with counting the limit by one varible, then by the other one. So

a) the existence one of the limits

$$\lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) \quad \text{and} \quad \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right)$$

not follows from the existence of the other limit.

b) If both previous mentioned limits exist, they can be different,

c) moreover if the above mentioned limits are equal, it is not necessary to exist the limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y).$$

Now we consider some examples for the previous propositions.

Example 1.

a) Let $f(x,y) = \sin\frac{x}{y}$, where $x, y \neq 0$. Hence

$$\lim_{x\to 0}\sin\frac{x}{y} = 0, \quad \text{so} \quad \lim_{y\to 0}\left(\lim_{x\to 0}\sin\frac{x}{y}\right) = 0.$$

But the limit

$$\lim_{y\to 0}\sin\frac{x}{y}$$

does not exit, so the limit

$$\lim_{x\to 0}\left(\lim_{y\to 0}\sin\frac{x}{y}\right)$$

neither.

b) If $f(x,y) = \frac{x^2+y^2+x-y}{x+y}$ then

$$\lim_{x\to 0} f(x,y) = y-1, \quad \text{so} \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = -1,$$

while

$$\lim_{y\to 0} f(x,y) = x+1, \quad \text{so} \quad \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 1.$$

c) In the case of $f(x,y) = \left|\frac{x-y}{x+y}\right|$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\left|\frac{-y}{y}\right| = 1, \quad \text{so} \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 1.$$

Similarly

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\left|\frac{x}{x}\right| = 1, \quad \text{so} \quad \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 1,$$

but the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist. Namely for example if

$$x = y \to 0, \quad \text{then} \quad f(x,y) \to 0,$$

while if

$$x = 3y \to 0, \quad \text{then} \quad f(x,y) \to \frac{1}{2},$$

it means that the limit of f at $(0,0)$ does not exist.

Definition 2. We say that the function $f(x,y)$ is *continous at* $P_0(x_0,y_0) \in D_f$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

If f is contionus at each point of its domain D_f, we say that the f is *continous.*

Example 2. Consider the function

$$f(x,y) = \begin{cases} y\sin\frac{1}{x}, & \text{if} \quad x \neq 0, \quad y \in \mathbb{R} \\ 0, & \text{if} \quad x = 0, \quad y \in \mathbb{R}. \end{cases}$$

The function $f(x,y)$ is not continous at the points $(0,y)$, where $y \neq 0$, because the limit of the function does not exist at these points. But

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0,$$

hence f is continous at the origin.

Example 3. The function $f(x,y) = x\sin y$ is continous for every $(x_0,y_0) \in \mathbb{R}^2$, since f is product of continous functions.