

What does the limit of a function of two variables at a point mean?

One dictionary defines a limit as something that can be "approached but never reached". But of course this is not a correct definition. In the following we give the limit definition of a function of two variables

Definition 1. Let $f : T \rightarrow \mathbb{R}$ be a function, such that $T \subset \mathbb{R}^2$ and suppose $P_0(x_0, y_0) \in \mathbb{R}^2$ is such that every open ball centered at $P_0(x_0, y_0)$ meets the domain T . If $A \in \mathbb{R}$ is such that for every $\varepsilon > 0$ there exists a real number $\delta > 0$ so that

$$|f(x, y) - A| < \varepsilon$$

whenever

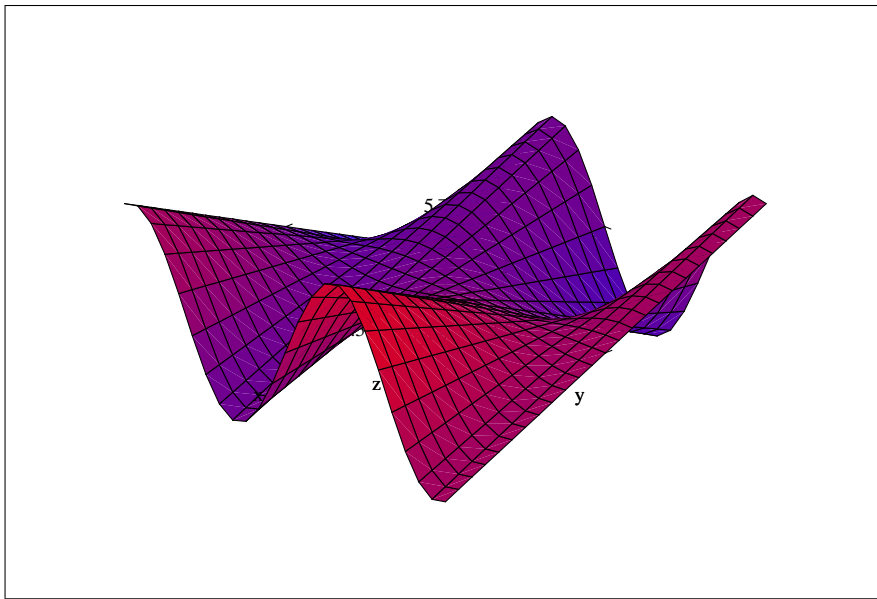
$$0 < (x - x_0)^2 + (y - y_0)^2 < \delta^2,$$

then we say that A is the *limit of f at P_0* .

Example 1. To illustrate it, consider the function

$$f(x, y) = x \sin y.$$

This function can be represented by surface $z = x \sin y$. We obtain the following compute-generated graph of this surface:

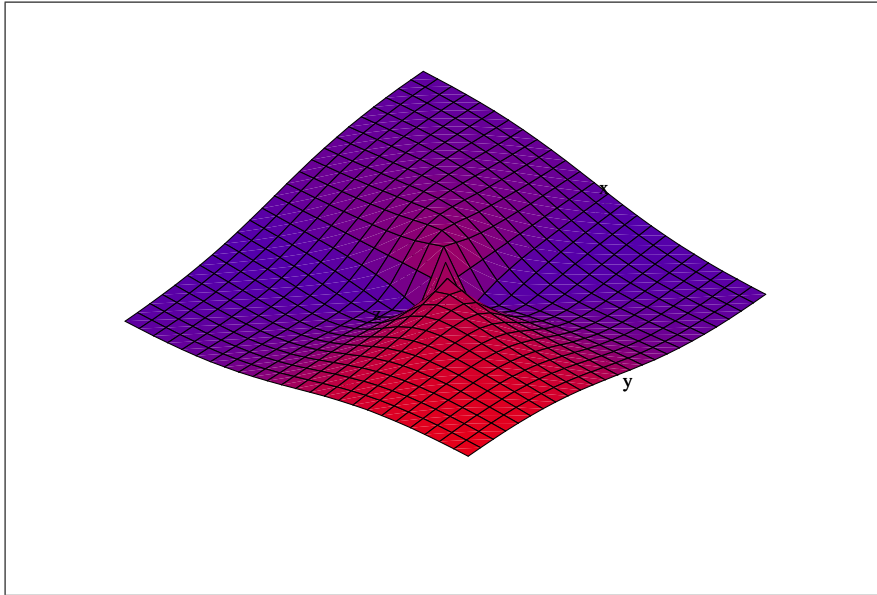


E. g. $\lim_{(x,y) \rightarrow (1,0)} f(x, y) = 1$, $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$

Example 2. The graph of

$$z = \frac{xy}{x + y}$$

is shown in figure:



If we consider the corresponding function $f(x, y) = \frac{xy}{x+y}$, then we see that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0,$$

but the value of $f(x, y)$ does not exist.

There are functions such that at some points the limit does not exist or exist, but not equal to the value of the function at the point. These cases motivate the following definition of the continuity.

Definition 2. We say that the function $f(x, y)$ is *continuous* at $P_0(x_0, y_0) \in D_f$ if

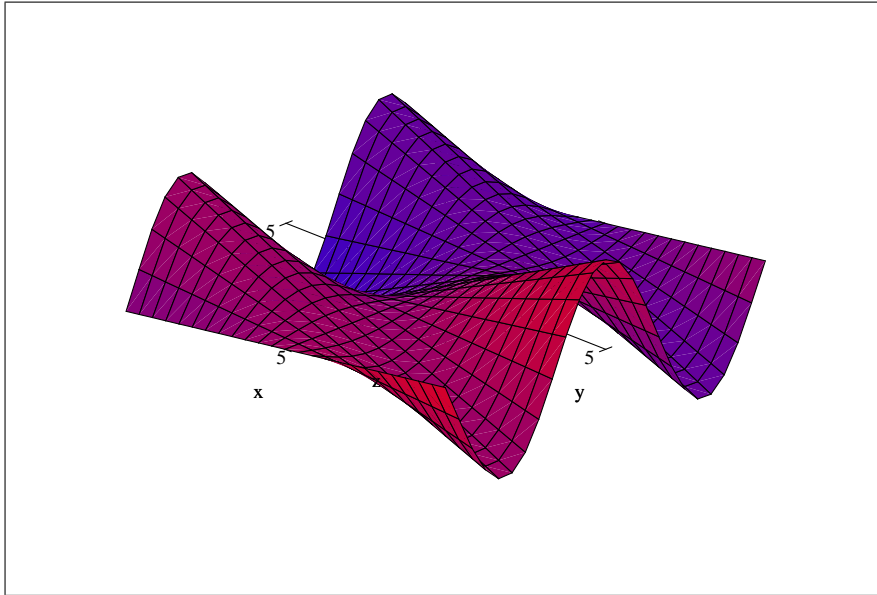
$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0).$$

If f is continuous at each point of its domain D_f , we say that the f is *continuous*.

Example 3. The function

$$f(x, y) = y \cos x$$

is continuous for every $(x_0, y_0) \in \mathbb{R}^2$, since f is product of continuous functions. The graph of the corresponding surface is shown in the following figure.



(Intuitively it means there are no "holes" on the surface.)