For the user

Financial mathematics is part of the compulsory courses in Mathematics and it focuses on linear programming and compounding interests. This part starts with mathematical sequences and series, because this information is needed when studying recurring payments related to compounding interest. This section is intended to be useful for all the students of financial mathematics so that they would get the basic abilities needed in management accounting and financing. This section has been planned so that the examples used could be facing us in every day life. Due to this the section is useful for all those who want to be able to make investment analysis and loan computations and to learn to make decisions of one’s own finances based on these.

This material is made especially for independent learning. From the table of contents of this material you will find a link to preliminary knowledge required for each chapter which you should know to be able to take in this information. The theoretical part of the material is clarified with examples. These are divided into two different categories for those studying independently; examples and exercises. You can find the solutions for the examples after the examples. Exercises are made for you to solve and they don’t include the solutions, instead with exercises you can find a link to page where you can find out if your answer is correct or not. Try to solve as many exercises as can because practicing is the best way to learn. Try to at least think how this exercise should be solved; it’s no very useful just to check out the solution. After each sub-chapter you can find a link to page where you can find more exercises related to the subject. There you can also find a link to a specific page which contains the solutions to these exercises. Each sub-chapter ends with a link to goals which you should know after studying this section. From the material you can find links to exercises to test your knowledge which include questions or mathematical problems about the subject. Please do remember practise makes perfect! This material supporting independent studying tries to offer a possibility to active learning, where you can practise what you have learned through exercises and at the same time you can observe what you have learned.

This material is based on unpublished work done at course of financial mathematics in polytechnic school. If you have any comments about it I wish to reserve those via e-mail. Please inform me if you find any faults in the material. Your feedback will be used to improve this material.

Enjoy studying financial mathematics!

2.1. Accrued and initial capital

After studying this section you should know how to

- evaluate the accrued capital when initial capital, interest rate and time are known,
- define the initial capital when the accrued capital, interest rate and time are known and
- **Apply calculating of the initial capital to comparison of the different options for payment.**

2.1.1. Accrued capital

The interest rate is added to the capital after every interest period. During the next period also this interest accumulates. First we learn how to define the accrued capital in a case where the interest rate of the capital accumulates for several periods. We use the following symbols:

k = initial capital
p = interest rate (%/interest period) and
K = accrued capital

Determine the grown capital in the end of every interest period. Let's use subscripts to inform after which interest period the grown capital in question has been evaluated. Let's evaluate first K_1 which is the accrued capital after the first interest period. The interest rate p %/interest period announces how many per cents the interest rate of the interest period is from the capital. When the interest rate of the first interest period is attached to the initial capital is the accrued capital K_1:

$$K_1 = k + \frac{p}{100} \cdot k.$$

When the initial capital k is taken as a common factor from this expression you will have the accrued capital in a form of

$$K_1 = \left(1 + \frac{p}{100}\right) \cdot k$$

so during one interest period the capital accrues $(1 + p/100)$ -fold.

Let's use this information to form a table where you can follow the increase of the capital after several interest periods.

After the 1. interest period: $K_1 = k + \frac{p}{100} \cdot k = \left(1+\frac{p}{100}\right) \cdot$

After the 2. interest period: $K_2 = \left(1+\frac{p}{100}\right) \cdot \left(1+\frac{p}{100}\right) \cdot k = \left(1+\frac{p}{100}\right)^2 \cdot k$

After the 3. interest period: $K_3 = \left(1+\frac{p}{100}\right) \cdot \left(1+\frac{p}{100}\right)^2 \cdot k = \left(1+\frac{p}{100}\right)^3 \cdot k$

After the 4. interest period: $K_4 = \left(1+\frac{p}{100}\right) \cdot \left(1+\frac{p}{100}\right)^3 \cdot k = \left(1+\frac{p}{100}\right)^4 \cdot k$

When this is continued it can be seen that after n interest periods the accrued capital is

$$K_n = \left(1 + \frac{p}{100}\right)^n \cdot k.$$

So when the initial capital k has accumulated with the interest rate p %/during interest period n, the accrued capital is K_n:

$$K_n = \left(1 + \frac{p}{100}\right)^n \cdot k.$$

Practise the usage the conducted solution via examples.

Example 2.1.

Calculate the accrued capital when 2000 euros are deposited in an account for 5 years with the interest rate of 2% per annum. In the account the interest will be added to the capital always after one year.

Solution:

The accrued capital after five years.

$$\left(1 + \frac{2}{100}\right)^5 \cdot 2000\ € = 1{,}02^5 \cdot 2000\ € = 2208{,}16\ €.$$

Answer: 2208,16 euros

Example 2.2.

Grandparents deposited 15 000 mk for their grandchild in 26th of April 1980 in an tax-free account which had the interest rate of 5,5% per annum. The interest was added to the capital in the end of every year. The account was closed 26th of April 1990 and all the money in there was drawn out from the account. How much money there was in the account at that time when it is assumed that the interest rate was the same during the whole time and no money was drawn out from the account during that period? The German way of calculating the interest period is used (30 days in a month, 360 days in a year).

Solution:

First has to be calculated how much the interest has accumulated by the end of year 1980. The interest period is 26.4.1980 - 31.12.1980.

$4+8 \cdot 30$ days = 244 days

The accumulated interest was

$$15000\ \text{mk} \cdot \frac{5{,}5}{100} \cdot \frac{244}{360} = 559{,}17\ \text{mk}.$$

In the end of the year 1980 the accrued capital was 15559,17 mk. By the end of the year 1989 this balance had accumulated in all together 9 interest periods. The accrued balance was 31st of December 1989

$$\left(1 + \frac{5{,}5}{100}\right)^9 \cdot 15559{,}17\ \text{mk} = 1{,}055^9 \cdot 15559{,}17\ \text{mk} = 25191{,}76\ \text{mk}.$$

Calculate the interest rate for this balance during the period between 31.12.1989 and 26.4.1990,

$3 \cdot 30 + 26$ days = 116 days

The accumulated interest was

$$25191{,}76\ \text{mk} \cdot \frac{5{,}5}{100} \cdot \frac{116}{360} = 446{,}45\ \text{mk}$$

and the accrued capital was

$$25191,76 \text{ mk} + 446,45 \text{ mk} = 25638,21 \text{ mk}.$$

Answer:

The balance of this account at 26th of April 1990 was 25638,21 mk, the interest included.

Exercise 2.3.

Calculate the accrued capital when 5800 euros are deposited for four years in an account which has the interest rate of tax at the source 2,75% per annum. The interest is added to the balance in the end of every year. Let's assume that the tax at source is 29% during the whole time. (Tip: calculate the net interest rate by taking away the tax at source from the gross interest rate.)

Exercise 2.3

Net interest rate is

$0,71 \cdot 2,75\% = 1,9525\%$

The accrued capital is then

$$\left(1 + \frac{1,9525}{100}\right)^4 \cdot 5800 € = 1,019525^4 \cdot 5800 € = 6266,42 €.$$

Answer: 6266,42€

2.1.2. The initial capital

Let's examine then an inverse problem. Which initial capital k will accumulate during interest period t n according interest rate p %/ to be the accrued capital K? This initial capital k can be solved with the previous compounding interest formula

$$K = \left(1 + \frac{p}{100}\right)^n \cdot k.$$

From this equation the initial capital k can be solved by dividing the equation by half by the coefficient of this unknown $(1 + p/100)^n$, and you will get

$$k = \frac{K}{\left(1 + \frac{p}{100}\right)^n}.$$

The initial balance has to be substituted

$$k = \frac{K}{\left(1 + \frac{p}{100}\right)^n},$$

so that during n interest period and with the interest rate p % per period you could get the accrued capital K.

Example 2.4.

What amount of money should be deposited in an account which has the interest rate of 2 % per annum so that with the interests after 18 years there would be 2500 euros? It is assumed that the interest rate will be unchangeable during this time and that the interest is added to the balance always after one year.

Solution:

The substitution is

$$\frac{2500\ €}{\left(1+\frac{2}{100}\right)^{18}} = \frac{2500\ €}{1{,}02^{18}} = 1750{,}40\ €.$$

Answer: 1750,40 euros has to be deposited in the account.

The defining of the initial capital is used when comparing different payment options. When these different options for payment include unequal amounts which are paid at different times and these can be made comparable by calculating the total value of each of them at the given time. For example; if the chosen time is at the conclusion of the sale is the value of the payments which are dueing later calculated at this moment by determining the initial capital congruent with the grown capital of the size of the payment.

Example 2.5.

Which total sum paid at the conclusion of the sale equates to next offer which is paid with the interest rate of 8,5% per annum:

57000 euros is paid at the conclusion of the sale and 43000 euros is paid after two years? Give the answer with the accuracy of ten euros.

Solution:

Calculate the initial value of the payment of 43000euros which falls due in two years:

$$\frac{43000\ €}{\left(1+\frac{8{,}5}{100}\right)^{2}} = \frac{43000\ €}{1{,}085^{2}} = 36526{,}58\ €.$$

The value of the offer at the time of the sale is

$$57000\ € + 36526{,}58\ € = 93526{,}58\ €.$$

Answer: 93530 euros

Example 2.6.

A company has two different payment options for buying premises:

(1) 580000 euros at the conclusion of the sale

(2) 230000 euros at the conclusion of the sale, 150000 euros after one year and 210000 euros after two years.

Which option is more profitable for this company, when the interest rate used in this comparison is 8,5 %:n per annum?

Solution:

Calculate the value of each payment option at the conclusion of the sale.

The value of option (1) is of course 580000 euros.

The value of the option (2) is

$$230000 + \frac{150000}{1{,}085} + \frac{210000}{1{,}085^2} = 546634{,}46 \text{ €}.$$

Answer:

Payment option number two is more favourable for the company.

2.2. Recurring payments

Objectives

After studying this section you should know how to

- define the terminal value of recurring payments when payments take place in the beginning or in the end of the interest period,
- define the terminal value of recurring payments when payments take place several times during the interest period and
- define the initial value of the recurring payments by discounting their terminal value to the starting point.

In this section a situation where equal payments are paid in regular intervals. You will learn how to calculate the common value of these payments with interests in a given time. The methods of calculation related to recurring payments are based on finite mathematical sequences and series. If necessary revise the geometric and arithmetic sequences and series before this section. Because this section is basically just about the applications of the sequences it will concentrate on examining calculations connected to the recurring payments by examples.

2.2.1. The common terminal value of the recurring payments

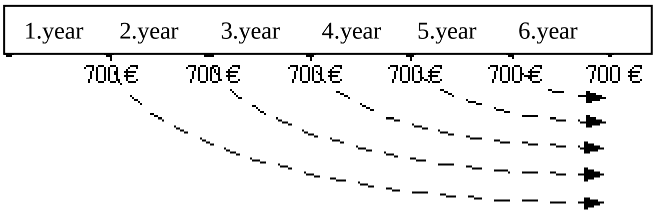
Practice first how to determine the common terminal value of the equal payments paid in regular intervals.

Example 2.7.

Matti deposits 700 euros in a tax-free account, which has the interest rate of 2% per annum, in the end of every year during six years. The interest rate is added to the balance in the end of each year. What is the balance of the account after the last deposit?

Solution:

The following figure illustrates the situation.



First determine the value of each deposit (the accrued balance) in the end of sixth year.

6. deposit: 700 €

5. deposit: $1{,}02 \cdot 700\ €$ (accumulated interest for one year)

4. deposit: $1{,}02^2 \cdot 700\ €$ (accumulated interest for two years)

3. deposit: $1{,}02^3 \cdot 700\ €$ (accumulated interest for three years)

2. deposit: $1{,}02^4 \cdot 700\ €$ (accumulated interest for four years)

1. deposit: $1{,}02^5 \cdot 700\ €$ (accumulated interest for five years)

The grown balances consisting of the deposits form a finite 6-membered geometric sequence where the first member is 700 euros and the ratio of two successive members is 1,02. Due to this the terminal value of the recurring payments is a geometric series

$a = 700\ €$ (1. addend),

$q = 1{,}02$ (ratio) and

$n = 6$ (number of the terms).

The combined terminal value of the deposits by the formula of geometrical series is

$$\begin{aligned} &700\ € + 1{,}02 \cdot 700\ € + \ldots + 1{,}02^5 \cdot 700\ € \\ &= \frac{700\ € \cdot (1 - 1{,}02^6)}{1 - 1{,}02} = 4415{,}68\ €. \end{aligned}$$

Answer:

The balance of the account is 4415,68 euros.

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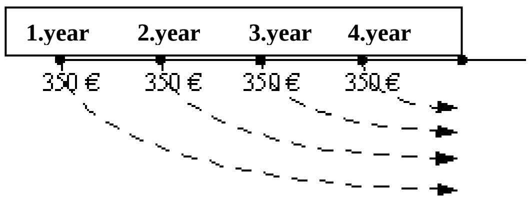
Exercise 2.8.

Liisa deposited in the beginning of every year, in four successive years, 350 euros into a tax at source account with interest rate 1,75% per annum. What was the balance of the account in the end of the fourth year, when the tax at source was 29% all this time? The interest is added to the capital in the end of each year. NB: calculate the net interest rate of the account first.

Exercise 2.8

The case is illustrated by a graph.



First calculate seperately the terminal value of each deposit in the end of the fourth year. Determine the net interest rate of the account to be able to calculate the terminal values. You will get that by subtracting the tax at source from the gross interest rate. The net interest rate is

0,71 · 1,75%/per annum = 1,2425%/per annum

The terminal value of the deposits:

4. deposit: $1{,}012425 \cdot 350\ €$ **(accumulated interest for one year)**

3. deposit: $1{,}012425^2 \cdot 350\ €$ **(accumulated interest for two years)**

2. deposit: $1{,}012425^3 \cdot 350\ €$ **(accumulated interest for three years)**

1. deposit: $1{,}012425^4 \cdot 350\ €$ **(accumulated interest for four years)**

The accrued capitals consisting of the deposits form 4-membered geometric sequence where the first member is $1{,}012425 \cdot 350\ €$ and the ratio of two successive members is 1,012425. Due to this the terminal value of the recurring payments is a geometric series where

$a = 1{,}012425 \cdot 350\ €$ **(1. addend),**

$q = 1{,}012425$ **(ratio) and**

$n = 4$ **(number of the terms).**

The combined terminal value of the deposits by the formula of geometrical series is

$$\frac{1{,}012425 \cdot 350\ € \cdot (1 - 1{,}012425^4)}{1 - 1{,}012425} = 1444{,}03\ €.$$

Answer: There is 1444,03 euros in the account.

In the previous examples the payment took place either in the beginning or in the end of the interest period. It is possible that payments are done also several times during the interest period. Still interest is added to the capital in the end of the interest period in which case the accumulated interest has to be calculated using the simple interest calculation. If this happens during several interest periods the recurring payment is considered to be the payments done during the given period and the sum of those interests that are beard in the end of the interest period. This will be examined via the following examples.

Example 2.9.

Pirjo deposits 90 euros for seventeen years in the end of every month into a tax at source account with interest rate 1,25% per annum. What is the balance of the account in the end the seventeenth year? The interest is added to the capital in the end of every year. It is assumed that the interest rate and the tax at source stay unchangeable during this period.

Solution:

First determine the net interest rate of the account which is

0,71 · 1,25%/per annum = 0,8875

Calculate how much the deposits done during the year accumulate interests all together. Study first the deposit of each month separately. The deposit done in January accumulates interest for eleven months, the deposit made in February accumulates interest for 10 months etc. The accumulated interest for the deposits during one year can be calculated by thinking how many interest months will there be for the deposit of 90 euros all together. The number of interest months is

11+ 10 + 9 + … +1 + 0 m.

The addends form an arithmetic sequence, the first member of it is 11, the difference of two successive members is -1 and the number of terms is 12. The sum of these is calculated by the formula of arithmetic series which is the mean value of the first and last term multiplied by the number of addends

$$\frac{12 \cdot (11+0)}{2} m = 66m$$

The accumulated interest for the deposits in the end of the year is

$$90\,€ \cdot \frac{0{,}8875}{100} \cdot \frac{66}{12} = 4{,}393125\,€.$$

When the interest is added to the capital is the new recurring payment in the end of the year always

$$12 \cdot 90\,€ + 4{,}393125\,€ = 1084{,}393125\,€.$$

Then calculate the terminal value of the recurring payments in the end of the seventeenth year when in the end of each year a deposit of 1084,393125 euros is done to the account. Determine first the value of each deposit in the end of the seventeenth year.

17. deposit: 1084,393125 i

16. deposit: $1{,}008875 \cdot 1084{,}393125\,€$ (accumulated interest for one year)

15. deposit: $1{,}008875^2 \cdot 1084{,}393125\,€$ (accumulated interest for two years)

14. deposit: $1{,}008875^3 \cdot 1084{,}393125\,€$ (accumulated interest for three years)

$\dots$

1. deposit: $1{,}008875^{16} \cdot 1084{,}393125\,€$ (accumulated interest for 16 years)

The accrued capitals consisting of the deposits form an infinite 17-membered geometric sequence where the first member is 1084,393125 euros and the ratio of two successive members is 1,008875. Due to this the terminal value of the recurring payments is a geometric series where

$a = 1084{,}393125\,€$ (1. addend),

$q = 1{,}008875$ (ratio) and

$n = 17$ (number of the terms).

The combined terminal value of the deposits by the formula of geometrical series is

$$\frac{1084{,}393125\,€ \cdot (1 - 1{,}008875^{17})}{1 - 1{,}008875} = 19803{,}47\,€$$

Answer:

The balance of the account is19803,47 euros.

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Exercise 2.10.

Veli-Matti deposited 670 euros in the end of every march, june, september and december during five successive years into a tax at source account with interest rate 2,25% per annum. The interest is added to the capital in the end of each year. What was the balance of the account in the end of the fifth year, when the tax at source was 29% all this time?

Exercise 2.10

Solution:

The net interest rate of the account is 0,71 · 2,25%/per annum = 1,5975%/per annum. Calculate then how much the deposits during the year accumulate interest altogether. There are interest months for the deposits:

9 + 6 + 3 + 0 = 18 m

The accumulated interest for the deposits in the end of the year is

$$670\,€ \cdot \frac{1{,}5975}{100} \cdot \frac{18}{12} = 16{,}054875\,€.$$

When the interest is added to the capital is the new recurring payment done in the end of the year always

4 · 670€ + 16,054875€ = 2696, 054875€

Finally calculate combined terminal value of the recurring payments in the end of the fifth year when a payment of 2696,054875€ is done in the end of each year. The value of the individual in the end of the fifth year

5 .deposit: 2696,054875i

4. deposit: 1,015975 · 2696,054875€ (accumulated interest for one year)

3. deposit: $1{,}015975^2$ · 2696,054875€ (accumulated interest for two years)

2. deposit: $1{,}015975^3$ · 2696,054875€ (accumulated interest for three years)

1. deposit: $1{,}015975^4$ · 2696,054875€ (accumulated interest for four years)

The accrued capitals consisting of the deposits form a finite 5-membered geometric sequence where the first member is 2696,054875 euros and the ratio of two successive members is 1,015975. Due to this the terminal value of the recurring payments is a geometric series where

a = 2696,054875€ (first attend)

q = 1,015975 (ratio)

n = 5 (number of the terms)

The combined terminal value of the deposits by the formula of geometrical series is

$$\frac{2696{,}054875\,€ \cdot (1 - 1{,}015975^5)}{1 - 1{,}015975} = 13917{,}90\,€.$$

Answer:

There are 13917,90 euros in the account.

2.2.2. The combined initial value of the recurring payments

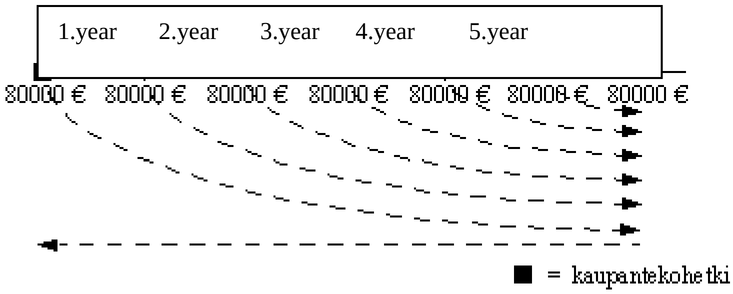
When comparing different kinds of recurring payments it is often necessary to define the initial value of the recurring payments in a certain starting point. The result tells what amount of money at the starting point is equal to recurring payments with the current interest rate. The Initial value of the recurring payments is calculated by first determining their combined terminal value (for instance at the time of the last payment) and by discounting this to the starting point. The discounting of the combined terminal value of the payments means defining that sum that accumulates according to the current interest rate to the size of the terminal value until the defining moment of terminal value. This was studied at the section of the defining the initial capital by compounding interest calculation. This will be examined via the following examples.

Example 2.11.

A company has agreed to pay the machinery they bought the following way: at the time of sale they pay 80 000 euros and after this 80 000 every year so that the first payment is done one year after the sale. Which payment done at the time of the sale these recurring payments correspond to when the imputed rate of the interest is 8,5 % per annum?

Solution:

Calculate the combined terminal value of these seven recurring payments. Determine first the value of each payment (accrued capital) at the time of the last payment six years after the deal.



7. payment : 80000 i

6. payment: $1{,}085 \cdot 80000$ € (accumulated interest for one year)

5. payment: $1{,}085^2 \cdot 80000$ € (accumulated interest for two years)

…

1. payment: $1{,}085^6 \cdot 80000$ € (accumulated interest for six years)

The grown capitals consisting of the deposits form a finite 7-membered geometric sequence where the first member is 80 000 euros and the ratio of two successive members is 1,085. Due to this the terminal value of the recurring payments is a geometric series where

$a = 80000$ € (first addend),

$q = 1{,}085$ (ratio) and

$n = 7$ (number of the terms).

The combined terminal value of the deposits by the formula of geometrical series is

$$\frac{80000\text{ €} \cdot (1 - 1{,}085^7)}{1 - 1{,}085} = 724839{,}76\text{ €}.$$

Discount this sum to the time of the deal; so calculate what initial capital will be 724 839,76 euros in six years time when the interest rate is 8,5% per annum. On the grounds of derived result in section the initial payment is

$$\frac{724839{,}76\text{ €}}{1{,}085^6} = 444286{,}97\text{ €}.$$

It is comparable to pay 444286,97 euros at the time of the deal or 80 000 euros at the time of the deal and then 80 000 every six years or pay 724839,76 euros six years after the deal.

Answer:

The cash value of the purchase money is 444286,97 euros.

NB!

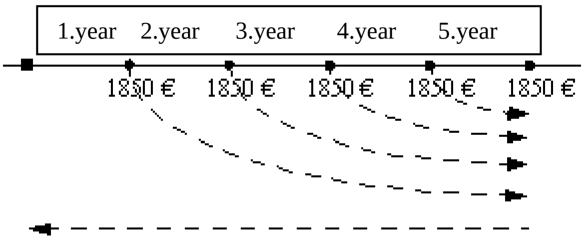
The cash value of the previous example can be calculated in other ways too. For example the initial value of each payment can be calculated at the time of the deal and these can then be summed up. The other way is to calculate separately the terminal value of the six payments done afterwards and the discount it to the time of the deal and add the payment done at the time of the deal to the initial payment that was just calculated. Calculate this example in these to different ways via spreadsheet so that you will get more exercise of the things taught in this section.

Exercise 2.12.

Otto has decided to deposit 1850 euros during five successive years into a tax at source account with interest rate 2,25% per annum. What amount of money should he deposit in the beginning of the first year so that it would give the same accrued capital? The tax at source is 29% during this period.

Solution:

The net interest rate of the account is $0{,}71 \cdot 2{,}25\%$ / per annum = 1,5975%/ per annum. This situation is illustrated with a graph.



Calculate the combined terminal value of the deposits in the end of the fifth year. The value of each separate deposit in the end of the fifth year is

5. deposit: 1850 €

4. deposit: $1{,}015975 \cdot 1850$ € (accumulated interest for one year)

3. deposit: $1{,}015975^2 \cdot 1850$ € (accumulated interest for two years)

2. deposit: $1{,}015975^3 \cdot 1850$ € (accumulated interest for three years)

1. deposit: $1{,}015975^4 \cdot 1850$ € (accumulated interest for four years)

The accrued capitals consisting of the deposits form a finite 5-membered geometric sequence where the first member is 1850 euros and the ratio of two successive members is1,015975 . Due to this the terminal value of the recurring payments is a geometric series where

$a = 1850$ € (first addend),

$q = 1{,}015975$ (ratio) and

$n = 5$ (number of the terms).

The combined terminal value of the deposits by the formula of geometrical series is

$$\frac{1850\text{ €} \cdot (1 - 1{,}015975^5)}{1 - 1{,}015975} = 9550{,}30\text{ €}.$$

Then determine which amount of money will accrue to this sum in five years time with the interest rate 1,5975 % per annum. The initial capital is

$$\frac{9550{,}30\text{ €}}{1{,}015975^5} = 8822{,}70\text{ €}.$$

Answer:

8822,70 euros should be deposited into the account at the beginning of the first year.

2.3. Constant payment loan

Objectives

After studying this section you should know how to

- calculate the constant payment of the constant payment loan when it is paid annually, semi-annually, quarterly or monthly,
- make a table of the instalments of the constant payment loan, that shows the interest and redemption of each constant payment,
- determine the last instalment if the constant payment is rounded up and
- Calculate the total interest when paying back the constant payment loan.

The invariable amount of the constant payment loan is paid back at each date of maturity. This constant payment includes the interest of the remaining loan and the rest of instalment is redemption. By decrease of the loan capital the proportion of the interests in the constant payment is diminishing and the proportion of the instalment accrues. In this section you will learn how to calculate the constant payment of the constant payment loan and how to make a table of redemptions of this loan where you can see the interest and instalment of each of the constant payments. In addition the total interests of the constant payment loan and loan with a fixed amortization schelude is being compared.

The constant payment of the reimbursement of the constant payment loan is determined so that the initial value of the paid constant payments with the interest rate agreed at the time of the borrowing is the size of the loan. No when you can define the initial common value of the recurring payments you can also calculate the amount of the constant payment. Let's start with the simplest case.

2.3.1. Annuity loan

First we will study the kind of constant payment loans where the instalment is paid early. These are called annuity loans. Let's make an expression to calculate the annuity in other words annual instalment. The following symbols are used:

N = loan (euros)

p = interest rate of the loan (%/per annum)

n = number of the payments

A = annual instalment (euros)

According to the loan programme the first constant payment A euros is paid one year after taking the loan. The second constant payment is paid after two years and so on. When the loan period is n years, the last constant payment n is paid after this time. Calculate first the combined terminal value of these recurring payments with the interest rate p %/ per annum and the time when the last constant payment is made; n years after the loan taking.

Determine first the value of each annual instalment (the accrued capital) n at the time when the annuity is paid n years after the loan taking.

n . payment : A

$(n-1)$. payment: $\left(1+\frac{p}{100}\right)\cdot A$ (accumulated interest for one year)

$(n-2)$. payment: $\left(1+\frac{p}{100}\right)^2\cdot A$ (accumulated interest for two years)

...

2. payment: $\left(1+\frac{p}{100}\right)^{n-2}\cdot A$ (accumulated interest $n-2$ years)

1. payment: $\left(1+\frac{p}{100}\right)^{n-1}\cdot A$ (accumulated interest $n-1$ years)

The accrued capital consisting of the constant payments form a finite n -membered geometric sequence where the first member is A and the ratio of two successive members is $1+p/100$.

Due to this the terminal value of the recurring payments is a geometric series where

$a = A$ (first addend),

$q = 1+\frac{p}{100}$ (ratio) and

$n = n$ (number of terms).

The combined terminal value of the constant payments by the formula of geometrical series is

$$\frac{A\cdot\left(1-\left(1+\frac{p}{100}\right)^n\right)}{1-\left(1+\frac{p}{100}\right)}=\frac{A\cdot\left(1-\left(1+\frac{p}{100}\right)^n\right)}{-\frac{p}{100}}=\frac{A\cdot\left(\left(1+\frac{p}{100}\right)^n-1\right)}{\frac{p}{100}}.$$

Discount this sum to the time of the borrowing. The initial capital with the interest rate p % per annum that accrues during n years to the size of the combined terminal value of the constant payments is

$$\frac{A\cdot\left(\left(1+\frac{p}{100}\right)^n-1\right)}{\frac{p}{100}}:\left(1+\frac{p}{100}\right)^n=\frac{A\cdot\left(\left(1+\frac{p}{100}\right)^n-1\right)}{\left(1+\frac{p}{100}\right)^n\cdot\frac{p}{100}}.$$

The constant payment is determined so that this combined initial value of the constant payments is the amount of the loan. Form an equation where the combined initial value of the constant payments is marked equal to the loan Neuros:

$$\frac{A\cdot\left(\left(1+\frac{p}{100}\right)^{n}-1\right)}{\left(1+\frac{p}{100}\right)^{n}\cdot\frac{p}{100}} = N.$$

Solve the constant payment A from this equation. First multiply the equation by half by the left denominator and then you will get

$$A\cdot\left(\left(1+\frac{p}{100}\right)^{n}-1\right) = N\cdot\left(1+\frac{p}{100}\right)^{n}\cdot\frac{p}{100}.$$

The constant payment A can be solved by dividing the equation by half by the coefficient of this unknown and you will get

$$A = \frac{N\cdot\left(1+\frac{p}{100}\right)^{n}\cdot\frac{p}{100}}{\left(\left(1+\frac{p}{100}\right)^{n}-1\right)}.$$

If the annuity loan is Neuros, the interest rate p % per annum and the loan period n years, so the constant payment of the loan paid early in other words annuity A is

$$A = \frac{\left(1+\frac{p}{100}\right)^{n}\cdot\frac{p}{100}}{\left(\left(1+\frac{p}{100}\right)^{n}-1\right)}\cdot N \text{ euros}$$

Example 2.13.

A family raises a mortgage of 75 000 euros for 15 years with the interest rate of 5,4% per annum. The mortgage with it's interest is paid back by the constant payments done early. Calculate this constant payment done yearly so to say the annual instalment. How much interest does the family have to pay from this mortgage altogether?

Solution:

Calculate the constant payment of the mortgage by the previous formula, when

N = 7500 €,

p = 5,4 (%/ per annum) and

n = 15

The constant payment is

$$\frac{1{,}054^{15}\cdot 0{,}054}{1{,}054^{15}-1}\cdot 75000\text{ €} = 7422{,}34\text{ €}.$$

The total interest is elicited by subtracting from the total amount paid which is

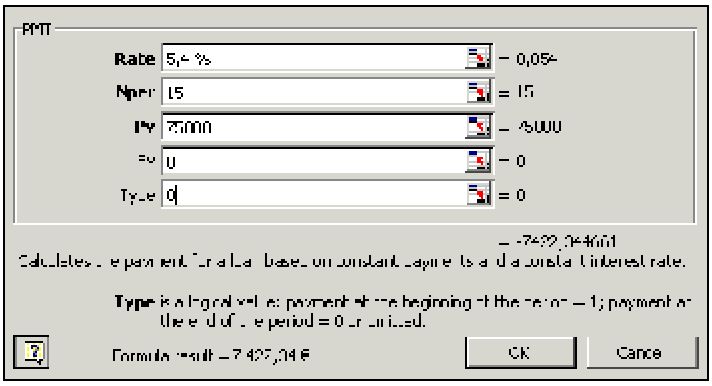
$$15\cdot 7422{,}34\text{ €} = 111335{,}10\text{ €},$$

The amount of the mortgage 75000 euros. Then the share of the interest from these payments is 36335,10 euros.

Answer: Annual instalment is 7422,34 euros. The total interest is 36335,10 euros.

The spreadsheets have normally an own function to calculate the constant payment of the annuity loan. By Excel-spreadsheet the constant payment of the loan can be calculated by function PMT(p %;n ;N ;0;0) in the English version and the function MAKSU(p %;n ;N ;0;0)in the Finnish version. The first argument of the function in the interest rate of the interest period that tells what is the share of the capital that is paid as interest during one interest period. The other argument tells what is the number of the constant payments which in the annuity loan is same as the loan period in years. The third argument is the initial value of the recurring payments, the amount of the loan. The fourth argument is the prospective value of the payment which is always zero when counting the constant payment of the loan. The fifth argument is number 0 or 1 depending on if the payments takes place in the end of the period (value 0) or in the beginning of it (value 1).   By calculating this way the result of the function will be a negative payment with the correct absolute value. If you want the constant payment in the table to be positive add minus sign in front of the PMT/MAKSU-function.

The annual instalment of the previous example can be calculated by the PMT –function of Excel by giving the following arguments:



From the graph you can see how the result of the actual PMT –function is negative. The result of the formula in the cell of the table is anyhow annuity as positive because I have set the formula to give the opposite number of the result of this PMT -function. The formula in the cell calculating the annual instalment is:

=-PMT(5,4 %;15;75000;0;0).

Next you will practice how to formulate a table of instalments of the constant payment loan. Then it is followed how the shares of the interest and instalments change in the constant payments. Each payment includes the interest from the loan that is left. The last part of the constant payment is instalment. If the constant payment is rounded off the last payment differs from the previous ones because then the rest of the loan is paid back with the interests.

Example 2.14.

The constant payment of the annuity loan of the previous example is rounded to the nearest euro. Make a table of instalments where the interest and the instalments of each payment can be seen. How big is the last instalment when the remaining loan and it’s interest is paid back? Calculate also the paid total interest.

Solution:

The constant payment rounded to the next euro is 7422 euros. Underlying is the table of instalments made by Excel-spreadsheet.

Payment	Loan	Annuity	Interest	Instalment
1	75 000,00€	7 422,00€	4 050,00€	3 372,00€
2	71 628,00€	7 422,00€	3 867,91€	3 554,09€
3	68 073,91€	7 422,00€	3 675,99€	3 748,01€
4	64 327,90€	7 422,00€	3 473,71€	3 948,29€
5	60 379,61€	7 422,00€	3 260,50€	4 161,50€
6	56 218,11€	7 422,00€	3 035,78€	4 386,22€
7	51 831,99€	7 422,00€	2 798,92€	4 623,08€
8	47 208,81€	7 422,00€	2 549,28€	4 872,72€
9	42 336,08€	7 422,00€	2 286,15€	5 135,85€
10	37 200,23€	7 422,00€	2 008,81€	5 413,19€
11	31 787,05€	7 422,00€	1 716,50€	5 705,50€
12	26 081,55€	7 422,00€	1 408,40€	6 013,60€
13	20 067,95€	7 422,00€	1 083,67€	6 338,33€
14	13 729,62€	7 422,00€	741,40€	6 680,60€
15	7 049,02€	7 429,67€	380,65€	7 049,02€
	In total	111 337,67€	36 337,67€	75 000,00€

In the credit column it is calculated how much is there left from the loan. The remainder of the loan can be calculated by subtracting the instalment included in the constant payment from the previous remainder of the loan. The annuity column contains the paid constant payment. The interest column contains the interest included in the constant payment that can be solved by calculating how much is 5,4% of the remaining loan because the payment period is one year aka one complete interest period. The instalment included in the constant payment is calculated by subtracting the interest from the paid constant payment to the instalment column. The last payment is calculated by summing up the remainder of the loan and it's interest from one year.

Answer:

The last payment is 7429,67 euros. The interest paid is 36337,67 euros all together.

Exercise 2.15.

Mister Lahtinen raises a loan of 25 200 euros for 8 years for investment purposes. The interest rate of the loan is 4,85% per annum.

(a) Make a table of instalments and calculate the total interest when the loan is paid yearly by equal instalment and in connection with payments also the interest for the remainder of the loan is paid.

(b) Calculate the annual instalment if the loan is paid back in equal annual payments. Make a table of instalments where one can see the interest and instalment of the constant payment when it is rounded to the nearest euro. How big is then the last payment? How much does mister Lahtinen have to pay interests in total?

Exercise 2.15

Solution:

a) Underlying is the table of instalments made by Excel-spreadsheet, including also the paid total interest.

Number of Payment	Loan	Interest	Instalment	Payment (€)
1	25 200,00€	1 222,20€	3 150,00€	4 372,20€
2	22 050,00€	1 069,43€	3 150,00€	4 219,43€
3	18 900,00€	916,65€	3 150,00€	4 066,65€
4	15 750,00€	763,88€	3 150,00€	3 913,88€
5	12 600,00€	611,10€	3 150,00€	3 761,10€
6	9 450,00€	458,33€	3 150,00€	3 608,33€
7	6 300,00€	305,55€	3 150,00€	3 455,55€
8	3 150,00€	152,78€	3 150,00€	3 302,78€
	In total	5 499,90€	25 200,00€	30 699,90€

b) Calculate first the annual instalment when *N* = 25 200€, *p* = 4,85%/per annum and *n* = 8. The annual instalment is

$$\frac{1{,}0485^{8}\cdot 0{,}0485}{1{,}0485^{8}-1}\cdot 25200\,€ = 3875{,}38\,€ \approx 3875\,€.$$

Then make a table of instalments where the interest and the instalment of the payment can be seen. Set up the last payment so that the remaining loan with it’s interest will be paid. Finally calculate the total interest.

Number of Payment	Loan	Annuity	Interest	Instalment
1	25 200,00€	3 875,00€	1 222,20€	2 652,80€
2	22 547,20€	3 875,00€	1 093,54€	2 781,46€
3	19 765,74€	3 875,00€	958,64€	2 196,36€
4	16 849,38€	3 875,00€	817,19€	3 057,81€
5	13 791,57€	3 875,00€	668,89€	3 206,11€
6	10 585,46€	3 875,00€	513,39€	3 361,61€
7	7 223,86€	3 875,00€	350,36€	3 524,64€
8	3 699,22€	3 878,63€	179,41€	3 699,22€
	In total	31 003,63€	5 803,63€	25 200,00€

Answer:

a) The total interest of the constant payment loan is 5 499,90€.

b) The annuity of the annual instalment loan is 3875 euros when the last annuity is 3878,63 euros. The total interest in the annual instalment loan is 5803,63 euros.

2.3.2. Constant payment loan with several constant payments during the interest period

A customer who has taken a bank loan usually wants to divide payments of the loan evenly around the year. The payments of the constant payment loans are usually paid semi-annually, quarterly or monthly. When the constant payment is calculated is the interest rate for the payment period calculated from the interest per annum the following way; the semi-annual interest rate is the annual interest rate divided to half, the quarterly interest rate is quarter of the annual interest rate and the monthly interest rate is the annual interest rate divided by twelve. The constant payment is calculated by the previous annuity formula where the annual interest rate is substituted by the interest rate of the payment period.

When

N = loan (€)

p = interest rate per annum

n = number of the payments

m = number of the payments per annum

is the constant payment A solved by the following formula:

$$A = \frac{\left(1+\frac{p}{100m}\right)^n \cdot \frac{p}{100m}}{\left(\left(1+\frac{p}{100m}\right)^n - 1\right)} \cdot N.$$

By Excel-spreadsheet you can easily calculate the constant payment also then when there are several constant payments during the interest period. The constant payment is then calculated by the same function than the annual instalment in the annuity loan, in the english version with the function PMT($(p\,\%)/m$;n ;N ;0;0) and in the Finnish version with the function PAYMENT($(p\,\%)/m$;n ;N ;0;0). The first argument of the function is the interest rate of the interest period that tells which amount of the capital is paid as interest during one interest period. The interest rate of the payment period can be calculated by dividing the annual interest rate $p\,\%$ by the number of payments in one year m . The other arguments of the function are the same as when calculating the annuity. Remember to put the opposite number of this function to the formula of calculating the constant payment if you want the result to be positive.

Example 2.16.

Family raises a mortgage of 109 800 euros for 15 years for buying a house. The annual interest rate of the mortgage is 4,9%. The mortgage is paid back with the constant payments done monthly. Calculate the constant payment and the interest rate. How much a family would have had to pay interest if instead of the constant payment loan they would have chosen monthly paid loan with a fixed amortization schelude?

Solution:

Calculate the constant payment by the previous formula when

N = 109 800 €

p = 4,9%/ per annum

n = 12 · 15 = 180 and

m = 12.

The constant payment is

$$\frac{\left(1+\frac{4,9}{100\cdot 12}\right)^{180}\cdot\frac{4,9}{100\cdot 12}}{\left(\left(1+\frac{4,9}{100\cdot 12}\right)^{180}-1\right)}\cdot 109800\text{ €} = 862,58\text{ €}.$$

The constant payment can also be calculated by using the formula of the Excel – spreadsheet:

=-PMT(4,9%/12;15*12;109800;0;0).

Altogether they will have to do repay from the loan

$$180\cdot 862,58\text{ €} = 155264,40\text{ €}.$$

When the amount of the loan is subtracted from this is the share of the interest

$$155264,40\text{ €} - 109800\text{ €} = 45464,40\text{ €}.$$

Calculate the total interest if the mortgage in question would be a loan with fixed amortization schelude. It is known that the interest of the payments of a loan with fixed amortization schelude form an arithmetic sequence so the total amount of the interest can be calculated as arithmetic series when the number of the payments and the interest of the first and last payment is known. There are 180 payments. The first interest is paid from the total amount of the mortgage from one month so it is

$$109800\text{ €}\cdot\frac{0,049}{12} = 448,35\text{ €}.$$

The last interest is paid from the monthly instalment which is

$$\frac{109800\text{ €}}{180} = 610\text{ €}.$$

The last interest is

$$610\text{ €}\cdot\frac{0,049}{12}$$

The total interest of the corresponding loan with a fixed amortization schelude is the mean value of the first and last payment multiplied by the number of payments, then

$$180\cdot\frac{\left(448,35\text{ €} + 610\text{ €}\cdot\frac{0,049}{12}\right)}{2} = 40575,68\text{ €}.$$

Answer:

The constant payment is 862,58 euros and the total interest is 45464,40 euros. If the mortgage would be a loan with fixed amortization schelude the family would have to pay interests in total 40575,68 euros.

Exercise 2.17.

Jouni raised a loan of 6300 euros for three years to buy a car. The annual interest rate of the loan is 8,55 %. The loan is paid back by quarterly constant payments. Calculate the constant payment and make a table of instalments where the interest included in the payments and the instalment can be seen. How much Jouni in total has to pay interests from this constant payment loan? How much smaller would the total interest be if he would have settled on a quarterly paid loan with fixed amortization schelude?

Solution:

First calculate the constant payment of the loan when the loan is $N = 6300$ euros, interest rate is

$p = 8{,}55\%/$ per annum, the number of payments is $n = 4 \cdot 3 = 12$ and the number of yearly payments is $m = 4$. The constant payment is

$$\frac{\left(1+\frac{8{,}55}{100 \cdot 4}\right)^{12} \cdot \frac{8{,}55}{100 \cdot 4}}{\left(\left(1+\frac{8{,}55}{100 \cdot 4}\right)^{12}-1\right)} \cdot 6300\,€ = 600{,}77\,€.$$

Then make a table of instalments that shows the interests included in the payments and the instalments. Calculate also the total interest to the table.

Payment	Loan	Constant Payment	Interest	Instalment
1	6 300,00€	600,77€	134,66€	466,11€
2	5 833,89€	600,77€	124,70€	476,07€
3	5 357,82€	600,77€	114,52€	486,25€
4	4 871,58€	600,77€	104,13€	496,64€
5	4 374,94€	600,77€	93,51€	507,26€
6	3 867,68€	600,77€	82,67€	518,10€
7	3 349,58€	600,77€	71,60€	529,17€
8	2 820,41€	600,77€	60,29€	540,48€
9	2 279,92€	600,77€	48,73€	552,04€
10	1 727,89€	600,77€	36,93€	563,84€
11	1 164,05€	600,77€	24,88€	575,89€
12	588,16€	600,74€	12,57€	588,16€
	In total	7 209,21€	909,21€	6 300,00€

From this quarterly paid constant payment loan Jouni has to pay interests in total 909, 21 euros. Calculate also the total interest if the loan in question would be a quarterly paid loan with fixed amortization schelude. The interests of the payments in a loan with fixed amortization schelude form an arithmetic sequence. The total amount of the interests is calculated as arithmetic series when the number of payments is 12 is the first interest

$$6300\,€ \cdot \frac{8{,}55}{100} \cdot \frac{1}{4}$$

and the last interest paid from the instalment

($6300\,€/12 = 525\,€$), is

$$525\,€ \cdot \frac{8{,}55}{100} \cdot \frac{1}{4}.$$

The total interest of a loan with fixed amortization schelude paid quarterly is

$$12 \cdot \frac{\left(6300\,€ \cdot \frac{8{,}55}{100} \cdot \frac{1}{4} + 525\,€ \cdot \frac{8{,}55}{100} \cdot \frac{1}{4}\right)}{2} = 875{,}31\,€.$$

Answer:

Quarterly paid constant payment is 600,77 euros. The total interest of the constant payment loan is 909,21 euros. If the loan is quarterly paid loan with a fixed amortization schelude the total interest would be 875,31 euros.