

SOME IDEAS ABOUT THE USE OF CAS FOR TEACHING DIFFERENTIAL CALCULUS OF SEVERAL VARIABLES

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1. Introduction

Over the past 20 years Spain has seen the advent of CASs (Computer Algebra Systems) as a teaching complement for the mathematical training of engineering students. In general, however, this has not been accompanied by necessary changes in teaching methods to obtain optimal benefits from this tool. What usually happens is that teachers merely use the computer to do exactly the same as can be done with a pen and paper. This leads to an overload of work, using the CAS in a mechanical fashion but not for actually teaching mathematics. The true purpose of CASs (that is, to offer a tool for doing and learning mathematics) is replaced thus by a situation in which students gain expertise in the CAS without understanding the mathematical issues being dealt with.

The purpose of this chapter is to demonstrate a more integrated and harmonious use of a CAS (*Maple*, *Mathematica* or any other) within daily teaching practice. This has been narrowed down to show the possibilities of using CASs for giving a course in Differential Calculus of Several Variables. The proposal is the result of our experience as maths teachers in engineering schools in several different Spanish Universities where either *Mathematica* or *Maple* are used.

2. The starting point

In Engineering Schools, Differential Calculus of Several Variables is generally taught after a single semester course on Elementary Calculus. The usual topics in this area are: Basic Notions of \mathbb{R}^2 and \mathbb{R}^3 , Limits, Directional and Partial Derivatives, Homogeneous Functions, Chain Rule, Implicit Differentiation, Inverse Function, Taylor's Formula, Maxima and Minima, Lagrange Multipliers.

Usually there are 30 hours of classes (both theoretical and practical) for teaching this content and it is assumed that the personal work of each student will match the same number of hours.

The aim is that students should be able to gain a reasonable mastery of the concepts, to acquire basic manual calculation skills, for later use in mathematical problems or in

applications in subjects specific to engineering, and moreover to know when and how to use a CAS for solving a mathematical problem.

Finally, it should be stressed that students should gain a basic knowledge of the CAS to be used (*Mathematica* or *Maple*, depending on the University).

Therefore, in view of the lack of teaching time, if the aims listed above are to be achieved a new methodological focus is required.

3. The teaching material

To carry through the methodological proposal, teaching material for a standard course on Calculus of Several Variables has been compiled. This material consists of a textbook that systematically includes theoretical concepts and proposed and solved problems (see García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 2002), together with a CD that includes problems solved with the help of a CAS.

A description of the material featured in the CD is presented below; in particular, that relative to Differential Calculus of Several Variables, and at the same time explain the strategy followed to compile it. It is noted that the CD includes two alternative directories with parallel contents: one facilitated using *Maple* and the other with *Mathematica*. Accordingly, the examples offered in this paper may be used in either of the two systems.

This teaching material gives exhaustive coverage on all the above issues and allows a differentiated use by the teacher as a function of the particular needs of each case. Based on the teacher's own criteria, the material can be used either as class material or as a guide for the personal work of the students, attending to the students' own characteristics and the time available for developing the different topics.

The present investigation should help to palliate the scarcity of material for the teaching of Calculus of Several Variables, at least in Spanish, as compared with the vast amount of material currently available for the teaching of Calculus of a Single Variable.

4. The methodological proposal

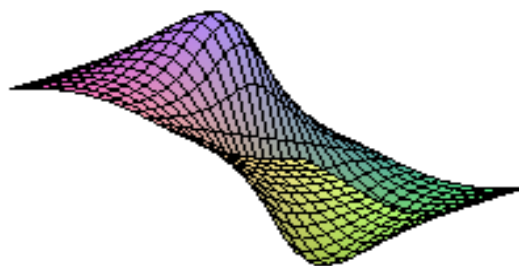
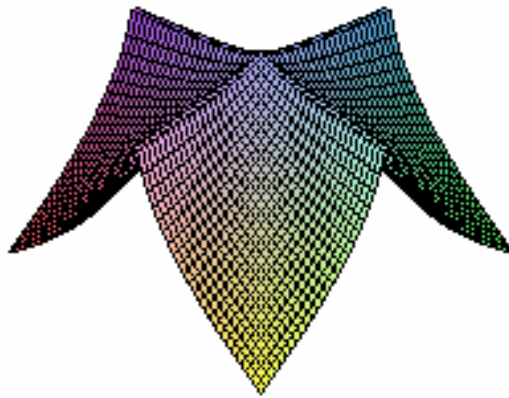
The methodological proposal is based on the integration of traditional teaching, lectures and problem classes (with the introduction of different concepts and blackboard exercises), with laboratory classes in which the corresponding CAS is used to reinforce the theoretical concepts and solve problems.

The methodological proposal offered involves the students dedicating approximately one third of the total work time programmed for the subject (combined class hours and

personal work) to guided use of the CAS, following the teacher's instructions and pursuing the following aims:

4.1. Improving visualisation

The possibility to sketch the graph of a surface immediately gives additional information in the study of a function of two variables, making conjectures about the properties of the function, e.g. extrema, bounds, etc. For example the next figures, both obtained with an elementary *Maple* input, help students to understand concepts such as non-differentiability and relative extrema.



The graphical capabilities of CAS can also be used by the teacher to offer presentations that will help introduce different mathematical concepts. However, it should be noted that complex programming is sometimes required to obtain “good” presentations of graphics and the teacher must take into account the cost of producing them.

4.2. Experimentation

The possibility of performing long tedious mathematical calculations in a short space of time enables the introduction of new strategies for problem solving, contrasting numerical, analytical and graphical techniques.

Thus, for example, with a CAS it is easy to know the variables that can be chosen as dependent in the implicit function theorem. To do so, assuming that the rest of the hypotheses of the theorem are satisfied, it is only necessary to check whether the determinant of the Jacobian matrix is null or not. So, even if the system of implicit functions is difficult, the problem is reduced to an automatic calculation.

Also, in order to get a first impression about the existence of a limit of a function it is possible to sketch a graph or construct tables of values for the function near the limit. For

example, to study the existence of the limit of the function $f(x, y) = \frac{xy}{x^2 + y^2}$ at the origin

of the coordinates, the Mathematica input

```
TableForm[Table[f[a + j h, b + k h], {j, 1, n}, {k, 1, n}]]
```

with $a=b=0$, $h=0.001$, $k=0.001$, $n=5$ generates the following table:

0.5	0.4	0.3	0.235294	0.192308
0.4	0.5	0.461538	0.4	0.344828
0.3	0.461538	0.5	0.48	0.441176
0.235294	0.4	0.48	0.5	0.487805
0.192308	0.344828	0.441176	0.487805	0.5

The dispersion of the data in this table leads one to suspect that the limit does not exist, which can be proved by using the technique of findings two subsets for which the limits will be different, for example, $y = x$ and $y=2x$.

As part of the teaching strategy, students should be encouraged to experiment when attempting to solve problems. A crucial aspect is to avoid laziness in the use of different trials since the mechanical processes are performed by the computer.

4.3. Release from mechanical work

The need for students to acquire basic computational skills should not be overlooked. Once it has been determined that students have assimilated the concepts and that they are able to do “operations” manually in cases of calculus that are not too complicated, they can then use CASs as “advanced calculators” for the more tedious calculations. Thus, for example, in the Chain Rule the Jacobian matrix of the function $g \circ f$ is the product of the Jacobian matrices of g and f . When the component functions of f and g have complicated

expressions, the Jacobian matrix of the function $g \circ f$, with $f: \mathbb{R}^5 \rightarrow \mathbb{R}^6$ and $g: \mathbb{R}^6 \rightarrow \mathbb{R}^4$ demands an enormous “manual” effort, which can be done automatically with a CAS.

Consequently, CASs, understood as “advanced calculators”, can be used in most topics of differential calculus.

The usefulness of such tools in intermediate calculations, relative to concepts that have already been analysed and that are necessary for later stages of learning, is also unquestionable. For example, the hypotheses of the existence of an inverse function can be analysed using a CAS, since the calculations involved in checking such hypotheses are routine. The use of a CAS therefore allows the user to save time during routine calculations which can be better used to model problems involving real situations and to interpret, at each step of the process, the results obtained.

As an example, the following problems are considered (taken from García, A., López, A., Romero, S., Rodríguez, G., de la Villa, A., 2002):

The temperature of a plate at any point (x,y) is given by the function $T(x,y) = 25 + 4x^2 - 4xy + y^2$. A heat alarm, situated at points on the circumference $x^2 + y^2 = 25$ is triggered at temperatures higher than 180°C or below 20°C . Will the alarm be triggered?

To solve this problem, students need to optimise $T(x,y)$ subject to the constraint $x^2 + y^2 = 25$. Then the method of Lagrange multipliers can be used. Calculations are made using the computer and the points

$$\{-2\sqrt{5}, \sqrt{5}\}, \{2\sqrt{5}, -\sqrt{5}\}, \{-\sqrt{5}, -2\sqrt{5}\}, \{\sqrt{5}, 2\sqrt{5}\}$$

are obtained as possible solutions. Then students must evaluate the function T at those points to find the extreme values and to interpret the result. Since the maximum of these values is lower than 180 and the minimum higher than 20 the alarm will not be triggered.

Moreover, the symbolic capacities of a CAS allow it to work efficiently with formal expressions, like those related to the properties between operators. For example it is trivial, using Mathematica, to prove $\text{div}(\text{grad } f) = \Delta f$. After loading the package **Calculus`VectorAnalysis`** the input

```
Div[Grad[f[x,y,z],Cartesian[x,y,z]]]-
Laplacian[f[x,y,z],Cartesian[x,y,z]]
```

produces the output 0.

Additionally, it is also possible to prove general properties in real problems as it is shown in the following example (taken from Marsden, J. and Weinstein, A., 1985):

The specific volume V , pressure P , and temperature T of a Van der Waals gas are related by the equation $P = \frac{RT}{V-b} + \frac{a}{V^2}$, where a , b , R are positive parameters. R is the universal constant of the gases, b represents the volume of the gas molecules in liquid state ($b \ll V$) and $\frac{a}{V^2}$ represents the inner pressure due to the interaction between molecules.

Students, using implicit function theorem, can prove that any two of the three variables V , P , or T may be considered independent in order to determine the third variable. Also they can find $\frac{\partial}{\partial P}T$, $\frac{\partial}{\partial V}P$ and $\frac{\partial}{\partial T}V$ and verify that $\left(\frac{\partial}{\partial P}T\right)\left(\frac{\partial}{\partial V}P\right)\left(\frac{\partial}{\partial T}V\right) = -1$

Here the CAS has been used for verifying the hypothesis of the implicit function theorem and for symbolic computations.

Accordingly, students can be asked for a generalisation for this formula when F is a function with n variables.

4.4. Distinction between algorithmic and non-algorithmic processes

The use of CASs is not a panacea that will allow the solution of any mathematical problem. In this sense, it is important to stress that students should distinguish between processes that are algorithmic and those that are not.

When a process can be “algorithmized”, students should be encouraged to perform a simple procedure consisting of translating the mathematical process to the language of the CAS used.

Within algorithmic processes, it is appropriate to distinguish between two types:

- Algorithms with an ensured answer, in which information is always obtained.

For example, it may be very simple for a student to perform a Maple procedure that will return the tangent plane at the surface, implicitly defined by the equation $F(x,y,z)=0$, at a point (a,b,c) . It suffices to know the definition and write the instructions for finding the gradient vector and the equation of the tangent plane. A basic form of this procedure could be as follows:

```
> Tang_Plane := proc (F, a, b, c)
  local gr;
  gr := subs({x=a, y=b, z=c}, linalg[grad](F(x, y, z), [x, y, z]));
  simplify(gr[1]*(x-a)+gr[2]*(y-b)+gr[3]*(z-c)=0)
end;
```

To use this procedure in later problems, the student must previously verify that the hypotheses of the Implicit Function Theorem are satisfied. The procedure can also be improved by including, inside it, the instructions necessary to see if such hypotheses are indeed satisfied and ensuring that an error message will appear if they are not. Also, it is possible to add a graphical instruction to draw the surface, together with the tangent plane. Students generally try to improve the procedure until it is as complete as possible.

- Algorithms that may include computational problems.

In this type of algorithm, the answer is conditioned to intermediate processes that cannot be performed owing to difficulties such as the impossibility of solving an equation, excessive systems requirements, etc.

Thus, for example, for the calculation of constrained extrema, the Lagrange multiplier algorithm requires the solution of a set of equations, which is not always guaranteed.

There are also non-algorithmic processes, for which interactive use of the CAS must be controlled. For example, finding limits is a non-algorithmic process.

It is possible to design alternatives using negative criteria such as reiterated limits or some positive criterion, such as the change to polar coordinates, which must be used with care (see next section). Thus, the function $f(x,y)$, after the changes $x = a + r \cos \phi$, $y = b + r \sin \phi$, is converted into the function $F(r, \phi)$. If this function has a uniform limit in the variable ϕ when $r \rightarrow 0$, then there exists a limit of $f(x,y)$ at the point (a, b) .

4.5. Fostering a critical spirit

It is clear that a CAS can easily facilitate the task of checking results. Students should, however, not develop a blind faith in the computer results, either intermediate or final; instead, they should at all times attempt to be in control of the process in the sense that the results should be compatible with the context of the problem that is being solved in each case. It should be noted that sloppy use of CASs may give rise to errors with which the student is unfamiliar, such as errors due to previous assignments in the work session.

Additionally, the limitations shown by CASs can help to foster students' critical awareness, as with the case when the answer given by the CAS is unexpected.

For example, the expected output of the Maple instruction

```
> mtaylor(sin(x*y)/(x*y), [x=0, y=0], 10);
```

should be $1 - \frac{x^2 y^2}{6} + \frac{x^4 y^4}{120}$ but the computer output is $1 - \frac{x^2 y^2}{6}$.

Moreover a CAS can produce a mistaken result due to program failure or due to the consideration of conditions on variables that the user has not taken into account (see Alonso, F., García, F., Hoya, S., Rodríguez, G., de la Villa, A., 2001).

For example, if, when calculating the limit at the point (1,1) of the function

$$f(x, y) = \frac{3x - 3x^2 + x^3 - 4y + 6y^2 - 4y^3 + y^4}{-2 + 3x - 3x^2 + x^3 + 4y - 6y^2 + 4y^3 - y^4}$$

A transformation to polar coordinates is used (see 4.4), where $a=b=1$, the resulting

$$\text{expression becomes } F(r, \theta) = \frac{\cos(\theta)^3 + r \sin(\theta)^4}{\cos(\theta)^3 - r \sin(\theta)^4}.$$

Mathematica and Maple simplify the limit of this expression to 1 when $r \rightarrow 0$. This implies that the limit does not depend on θ , but the implication is false, since if $\theta = \pi/2$, then the limit is -1 . Therefore, the double limit does not exist.

Students must see the need to control the results.

5. Conclusions

The purpose of this article is to propose a way of using mathematical software as a pedagogical tool. The graphical capacities of software packages can help to understand some concepts related to Differential Calculus of Several Variables.

Other possibilities of use are also shown: release of the mechanical work, possibility of experimentation, distinction between algorithmic and non-algorithmic processes and fostering a critical spirit about the obtained results. Additionally, it is necessary to address the importance of encouraging a change in the attitude of teachers to these kinds of topics.

Consequently, the software package chosen should not be more important as the way in which it is used; the same ideas and strategies with respect to software in education can be put into effect using different packages.

The possibilities of using new technologies are not limited to “witnessed” teaching (i.e., with the physical presence of the students); the teaching material compiled can be used in “virtual” teaching since there are no special problems involved in presenting the material online. In any case, this material can be completed with tutorials designed to facilitate its use.

REFERENCES

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