Stability of critical points in Linear Systems of Ordinary Differential Equations (SODE)

In this chapter *Mathematica* will be used to study the stability of the critical points of twin equation linear SODE.

If the SODE is linear and homogeneous $X' = AX$ with constant coefficients and the matrix $\boldsymbol{A}$ is non singular, then the only critical point is the origin and its stability can be studied by examining the eigenvalues of $\boldsymbol{A}$.

Example 1.

Compare the following SODE orbits and justify the different behaviour, analyzing the stability at the critical point.

a) $x' = 3x - y$, $y' = 2x + y$.

b) $x' = x + 5y$, $y' = -x - y$.

c) $x' = x - 5y$, $y' = -x - y$.

Solution

REMARK: The three SODE are autonomous, linear and with constant coefficients, with a single critical point at the origin.

Some trajectories are represented by phase planes.

Section a.

The SODE is defined and it is solved with the **DSolve** command

```
sis31a = {x'[t] == 3 x[t] - y[t], y'[t] == 2 x[t] + y[t]}
```

```
{x′[t] == 3 x[t] - y[t], y′[t] == 2 x[t] + y[t]}
```

```
solgen31a = DSolve[sis31a, {x[t], y[t]}, t]
```

$$\{\{x[t] \to -e^{2t} C[2] \text{Sin}[t] + e^{2t} C[1] (\text{Cos}[t] + \text{Sin}[t]),\ y[t] \to e^{2t} C[2] (\text{Cos}[t] - \text{Sin}[t]) + 2 e^{2t} C[1] \text{Sin}[t]\}\}$$

The solutions *x(t)* and *y(t)* are defined:

```
x[t_] = solgen31a[[1, 1, 2]]
```

$$-e^{2t} C[2] \text{Sin}[t] + e^{2t} C[1] (\text{Cos}[t] + \text{Sin}[t])$$

```
y[t_] = solgen31a[[1, 2, 2]]
```

$$e^{2t} C[2] (\text{Cos}[t] - \text{Sin}[t]) + 2 e^{2t} C[1] \text{Sin}[t]$$

Some solutions are generated with the instruction **Table**

```
solpar31a = Table[{x[t], y[t]} /. {C[1] -> i, C[2] -> j}, {i, -6, 6, 4}, {j, -6, 6, 4}]
```

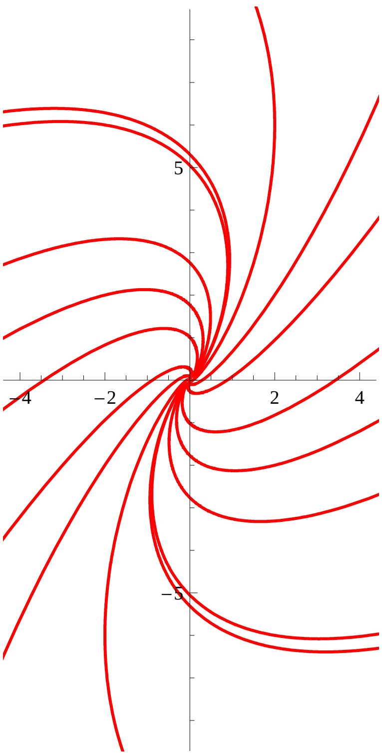
$$\{\{\{6 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\}\},$$
$$\{\{6 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\}\},$$
$$\{\{6 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\}\},$$
$$\{\{6 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\}\}\}$$

The previous table is modified with the command **Flatten** to make a functions list that is graphically represented.

```
solpar31abis = Flatten[solpar31a, 1]
```

$$\{\{6 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] - 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 12 e^{2t} \text{Sin}[t]\},$$
$$\{6 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] - 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) - 4 e^{2t} \text{Sin}[t]\},$$
$$\{6 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] + 2 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 4 e^{2t} \text{Sin}[t]\},$$
$$\{6 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\},$$
$$\{2 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), -2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\},$$
$$\{-2 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 2 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\},$$
$$\{-6 e^{2t} \text{Sin}[t] + 6 e^{2t} (\text{Cos}[t] + \text{Sin}[t]), 6 e^{2t} (\text{Cos}[t] - \text{Sin}[t]) + 12 e^{2t} \text{Sin}[t]\}\}$$

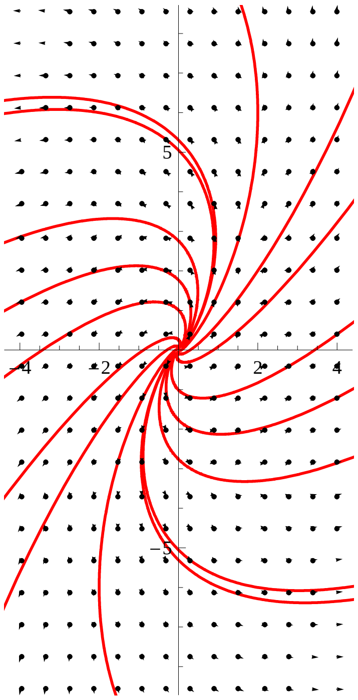
```
graf131a = ParametricPlot[Evaluate[solpar31abis],
  {t, -2, 0.5`}, PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.008`]}}]
```



The solutions and the directions field are represented together:

```
graf231a = (Needs["VectorFieldPlots`"];
   VectorFieldPlots`VectorFieldPlot[{3 x - y, 2 x + y}, {x, -15, 15}, {y, -20, 20},
    PlotPoints → 50, DisplayFunction → Identity, BaseStyle → {GrayLevel[0.2`]}]);
```

```
Show[graf131a, graf231a]
```



The trajectories move further from the origin (they spin like a spiral) as *t* increases and they are close to it when t is less than *0*. This behaviour can be explained by looking at the eigenvalues of the coefficient matrix.

The matrix associated with the SODE is defined as:

$$a = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

```
{{3, -1}, {2, 1}}
```

Its eigenvalues are given by:

```
Eigenvalues[a]
```

```
{2 + i, 2 - i}
```

The origin is an unstable spiral focus because the eigenvalues are complex conjugated and they have a positive real part.

Section b.

Any possible assignments are cleared and the SODE is defined:

```
Clear[x, y]
```

```
sis31b = {x'[t] == x[t] + 5 y[t], y'[t] == -x[t] - y[t]}
```

```
{x'[t] == x[t] + 5 y[t], y'[t] == -x[t] - y[t]}
```

It is solved using **DSolve** command:

```
solgen31b = DSolve[sis31b, {x[t], y[t]}, t]
```

$$\left\{\left\{x[t] \to \frac{5}{2} C[2] \operatorname{Sin}[2 t] + \frac{1}{2} C[1] (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]),\right.\right.$$
$$\left.\left. y[t] \to \frac{1}{2} C[2] (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - \frac{1}{2} C[1] \operatorname{Sin}[2 t]\right\}\right\}$$

The solutions *x(t)* and *y(t)* are defined:

```
x[t_] = solgen31b[[1, 1, 2]]
```

$$\frac{5}{2} C[2] \operatorname{Sin}[2 t] + \frac{1}{2} C[1] (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t])$$

```
y[t_] = solgen31b[[1, 2, 2]]
```

$$\frac{1}{2} C[2] (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - \frac{1}{2} C[1] \operatorname{Sin}[2 t]$$

Some solutions are generated with the instruction **Table**:

```
solpar31b = Table[{x[t], y[t]} /. {C[1] -> i, C[2] -> j}, {i, 2, 8, 3}, {j, 2, 8, 3}]
```

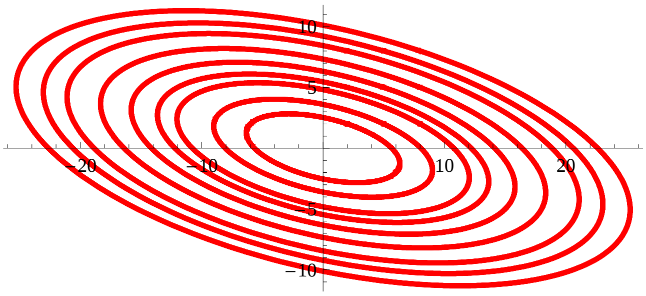
$$\Big\{\Big\{\{2 \operatorname{Cos}[2 t] + 6 \operatorname{Sin}[2 t], 2 \operatorname{Cos}[2 t] - 2 \operatorname{Sin}[2 t]\},$$
$$\Big\{2 \operatorname{Cos}[2 t] + \frac{27}{2} \operatorname{Sin}[2 t], \frac{5}{2} (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - \operatorname{Sin}[2 t]\Big\},$$
$$\{2 \operatorname{Cos}[2 t] + 21 \operatorname{Sin}[2 t], 4 (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - \operatorname{Sin}[2 t]\}\Big\},$$
$$\Big\{\Big\{5 \operatorname{Sin}[2 t] + \frac{5}{2} (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]), 2 \operatorname{Cos}[2 t] - \frac{7}{2} \operatorname{Sin}[2 t]\Big\},$$
$$\Big\{\frac{25}{2} \operatorname{Sin}[2 t] + \frac{5}{2} (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]), \frac{5}{2} (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - \frac{5}{2} \operatorname{Sin}[2 t]\Big\},$$
$$\Big\{20 \operatorname{Sin}[2 t] + \frac{5}{2} (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]), 4 (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - \frac{5}{2} \operatorname{Sin}[2 t]\Big\}\Big\},$$
$$\Big\{\{5 \operatorname{Sin}[2 t] + 4 (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]), 2 \operatorname{Cos}[2 t] - 5 \operatorname{Sin}[2 t]\},$$
$$\Big\{\frac{25}{2} \operatorname{Sin}[2 t] + 4 (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]), \frac{5}{2} (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - 4 \operatorname{Sin}[2 t]\Big\},$$
$$\{20 \operatorname{Sin}[2 t] + 4 (2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]), 4 (2 \operatorname{Cos}[2 t] - \operatorname{Sin}[2 t]) - 4 \operatorname{Sin}[2 t]\}\Big\}\Big\}$$

The previous table is modified with the command **Flatten** to make a functions list that is graphically represented.

```
solpar31bbis = Flatten[solpar31b, 1]
```

$$\left\{\{2\,\mathrm{Cos}[2\,t]+6\,\mathrm{Sin}[2\,t],\ 2\,\mathrm{Cos}[2\,t]-2\,\mathrm{Sin}[2\,t]\},\right.$$
$$\left\{2\,\mathrm{Cos}[2\,t]+\frac{27}{2}\mathrm{Sin}[2\,t],\ \frac{5}{2}(2\,\mathrm{Cos}[2\,t]-\mathrm{Sin}[2\,t])-\mathrm{Sin}[2\,t]\right\},$$
$$\{2\,\mathrm{Cos}[2\,t]+21\,\mathrm{Sin}[2\,t],\ 4\,(2\,\mathrm{Cos}[2\,t]-\mathrm{Sin}[2\,t])-\mathrm{Sin}[2\,t]\},$$
$$\left\{5\,\mathrm{Sin}[2\,t]+\frac{5}{2}(2\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t]),\ 2\,\mathrm{Cos}[2\,t]-\frac{7}{2}\mathrm{Sin}[2\,t]\right\},$$
$$\left\{\frac{25}{2}\mathrm{Sin}[2\,t]+\frac{5}{2}(2\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t]),\ \frac{5}{2}(2\,\mathrm{Cos}[2\,t]-\mathrm{Sin}[2\,t])-\frac{5}{2}\mathrm{Sin}[2\,t]\right\},$$
$$\left\{20\,\mathrm{Sin}[2\,t]+\frac{5}{2}(2\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t]),\ 4\,(2\,\mathrm{Cos}[2\,t]-\mathrm{Sin}[2\,t])-\frac{5}{2}\mathrm{Sin}[2\,t]\right\},$$
$$\{5\,\mathrm{Sin}[2\,t]+4\,(2\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t]),\ 2\,\mathrm{Cos}[2\,t]-5\,\mathrm{Sin}[2\,t]\},$$
$$\left\{\frac{25}{2}\mathrm{Sin}[2\,t]+4\,(2\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t]),\ \frac{5}{2}(2\,\mathrm{Cos}[2\,t]-\mathrm{Sin}[2\,t])-4\,\mathrm{Sin}[2\,t]\right\},$$
$$\left.\{20\,\mathrm{Sin}[2\,t]+4\,(2\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t]),\ 4\,(2\,\mathrm{Cos}[2\,t]-\mathrm{Sin}[2\,t])-4\,\mathrm{Sin}[2\,t]\}\right\}$$

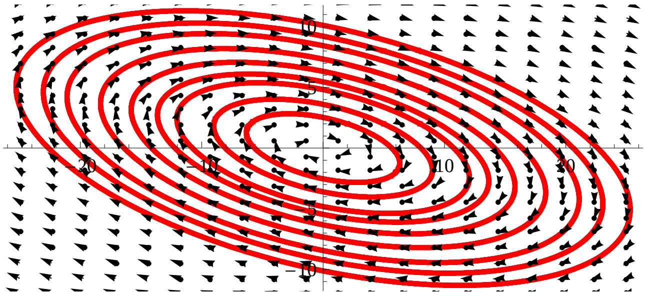
```
graf131b = ParametricPlot[Evaluate[solpar31bbis],
  {t, 0, 2 π}, PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.008`]}}]
```



The solutions and the direction fields are represented together:

```
graf231b = (Needs["VectorFieldPlots`"];
   VectorFieldPlots`VectorFieldPlot[{x + 5 y, -x - y}, {x, -25, 25}, {y, -12, 12},
    PlotPoints → 20, DisplayFunction → Identity, BaseStyle → {GrayLevel[0.2`]}]);
```

```
Show[graf131b, graf231b]
```



The trajectories are closed curves, in this case ellipses, that are centered in the origin. This behaviour can be explained by looking at the eigenvalues of the coefficient matrix.

The matrix associated with the SODE is defined as:

$$a = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}$$

```
{{1, 5}, {-1, -1}}
```

Its eigenvalues are given by the command:

```
Eigenvalues[a]
```

```
{2 i, -2 i}
```

The origin is a stable centre because the eigenvalues are pure imaginary.

Section c.

Any possible assignments are cleared and the SODE is defined:

```
Clear[x, y]
```

```
sis31c = {x'[t] == x[t] - 5 y[t], y'[t] == -x[t] - y[t]}
```

$\{x'[t] == x[t] - 5\,y[t],\ y'[t] == -x[t] - y[t]\}$

It is solved using the command:

```
solgen31c = DSolve[sis31c, {x[t], y[t]}, t]
```

$$\left\{\left\{x[t] \to \frac{1}{12}\, e^{-\sqrt{6}\,t}\left(6 - \sqrt{6} + 6\, e^{2\sqrt{6}\,t} + \sqrt{6}\, e^{2\sqrt{6}\,t}\right) C[1] - \frac{5\, e^{-\sqrt{6}\,t}\left(-1 + e^{2\sqrt{6}\,t}\right) C[2]}{2\sqrt{6}},\right.\right.$$
$$\left.\left. y[t] \to -\frac{e^{-\sqrt{6}\,t}\left(-1 + e^{2\sqrt{6}\,t}\right) C[1]}{2\sqrt{6}} - \frac{1}{12}\, e^{-\sqrt{6}\,t}\left(-6 - \sqrt{6} - 6\, e^{2\sqrt{6}\,t} + \sqrt{6}\, e^{2\sqrt{6}\,t}\right) C[2]\right\}\right\}$$

The solutions *x(t)* and *y(t)* are defined:

```
x[t_] = solgen31c[[1, 1, 2]]
```

$$\frac{1}{12}\, e^{-\sqrt{6}\,t}\left(6 - \sqrt{6} + 6\, e^{2\sqrt{6}\,t} + \sqrt{6}\, e^{2\sqrt{6}\,t}\right) C[1] - \frac{5\, e^{-\sqrt{6}\,t}\left(-1 + e^{2\sqrt{6}\,t}\right) C[2]}{2\sqrt{6}}$$

```
y[t_] = solgen31c[[1, 2, 2]]
```

$$-\frac{e^{-\sqrt{6}\,t}\left(-1 + e^{2\sqrt{6}\,t}\right) C[1]}{2\sqrt{6}} - \frac{1}{12}\, e^{-\sqrt{6}\,t}\left(-6 - \sqrt{6} - 6\, e^{2\sqrt{6}\,t} + \sqrt{6}\, e^{2\sqrt{6}\,t}\right) C[2]$$

Some solutions are generated with the instruction **Table**:

```
solpar31c =
 Simplify[Table[{x[t], y[t]} /. {C[1] -> i, C[2] -> j}, {i, 1, 5, 2}, {j, 1, 5, 2}]]
```

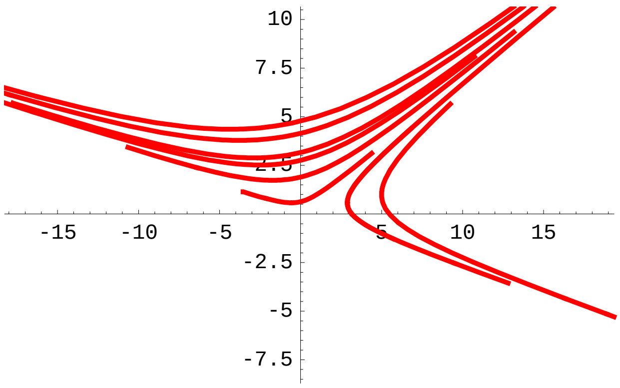
$$\left\{\left\{\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},\right.\right.$$
$$\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(3+7\sqrt{6}+\left(3-7\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(9+2\sqrt{6}+\left(9-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left.\left\{\frac{1}{2} e^{-\sqrt{6}\,t}\left(1+4\sqrt{6}+\left(1-4\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{2} e^{-\sqrt{6}\,t}\left(5+\sqrt{6}-\left(-5+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\}\right\},$$
$$\left\{\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(9+\sqrt{6}-\left(-9+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},\right.$$
$$\left\{\frac{1}{2} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{2} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left.\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(9+11\sqrt{6}+\left(9-11\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(15+4\sqrt{6}+\left(15-4\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\}\right\},$$
$$\left\{\left\{\frac{5}{2} e^{-\sqrt{6}\,t}\left(1+e^{2\sqrt{6}\,t}\right),\ \frac{1}{2} e^{-\sqrt{6}\,t}\left(1+\sqrt{6}-\left(-1+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},\right.$$
$$\left\{\frac{5}{6} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(9+4\sqrt{6}+\left(9-4\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left.\left.\left\{\frac{5}{6} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{5}{6} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\}\right\}\right\}$$

The previous table is modified with the command **Flatten** to make a functions list that is graphically represented.

```
solpar31cbis = Flatten[solpar31c, 1]
```

$$\left\{\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},\right.$$
$$\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(3+7\sqrt{6}+\left(3-7\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(9+2\sqrt{6}+\left(9-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left\{\frac{1}{2} e^{-\sqrt{6}\,t}\left(1+4\sqrt{6}+\left(1-4\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{2} e^{-\sqrt{6}\,t}\left(5+\sqrt{6}-\left(-5+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(9+\sqrt{6}-\left(-9+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left\{\frac{1}{2} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{2} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left\{\frac{1}{6} e^{-\sqrt{6}\,t}\left(9+11\sqrt{6}+\left(9-11\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(15+4\sqrt{6}+\left(15-4\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left\{\frac{5}{2} e^{-\sqrt{6}\,t}\left(1+e^{2\sqrt{6}\,t}\right),\ \frac{1}{2} e^{-\sqrt{6}\,t}\left(1+\sqrt{6}-\left(-1+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left\{\frac{5}{6} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{1}{6} e^{-\sqrt{6}\,t}\left(9+4\sqrt{6}+\left(9-4\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\},$$
$$\left.\left\{\frac{5}{6} e^{-\sqrt{6}\,t}\left(3+2\sqrt{6}+\left(3-2\sqrt{6}\right)e^{2\sqrt{6}\,t}\right),\ \frac{5}{6} e^{-\sqrt{6}\,t}\left(3+\sqrt{6}-\left(-3+\sqrt{6}\right)e^{2\sqrt{6}\,t}\right)\right\}\right\}$$

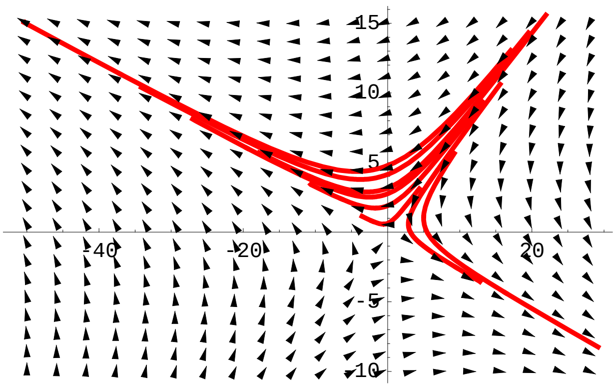
```
graf131c = ParametricPlot[Evaluate[solpar31cbis],
  {t, -0.5`, 1}, PlotStyle → {{RGBColor[1, 0, 0], Thickness[0.008`]}}]
```



The solutions and the direction fields are represented together:

```
graf231c = (Needs["VectorFieldPlots`"];
   VectorFieldPlots`VectorFieldPlot[{x - 5 y, -x - y}, {x, -50, 28}, {y, -10, 15},
    PlotPoints → 20, DisplayFunction → Identity, BaseStyle → {GrayLevel[0.2`]}]);
```

```
Show[graf131c, graf231c]
```



The eigenvalues of the matrix associated with the SODE are calculated to explain the nature of the point.

The matrix associated to the SODE is defined as:

```
a = (1   -5
     -1  -1)
```

{{1, -5}, {-1, -1}}

Its eigenvalues are given by the command:

```
Eigenvalues[a]
```

$\{-\sqrt{6}, \sqrt{6}\}$

The origin is a saddle point because the eigenvalues are real and with different sign.

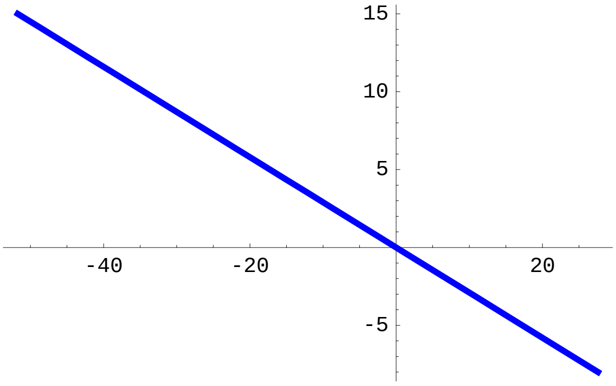
The eigenvectors are calculated and the straight lines with this directions are represented.

```
Eigenvectors[a]
```

$\{\{-1+\sqrt{6}, 1\}, \{-1-\sqrt{6}, 1\}\}$

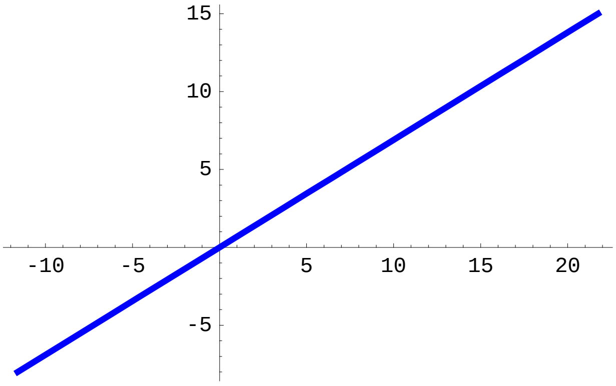
The straight line of the eigenvector corresponding to the negative eigenvalue is

```
graf331c = ParametricPlot[{(-1 - √6) t, t},
  {t, -8, 15}, PlotStyle → {RGBColor[0, 0, 1], Thickness[0.01`]}]
```



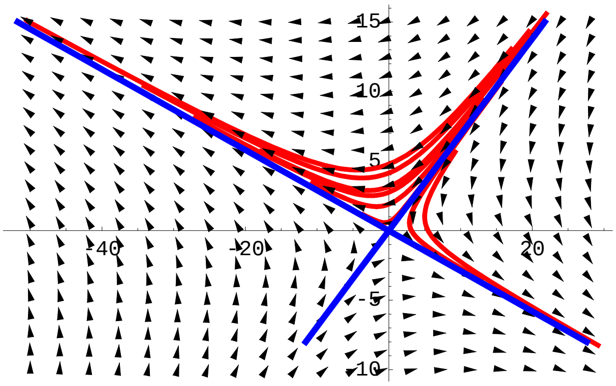
The straight line of the eigenvector corresponding to the positive eigenvalue is

```
graf431c = ParametricPlot[{(-1 + √6) t, t},
  {t, -8, 15}, PlotStyle → {RGBColor[0, 0, 1], Thickness[0.01`]}]
```



And the four graphs together:

```
Show[graf131c, graf231c, graf331c, graf431c]
```



It is possible to say that any trajectory that starts at a point on the straight line corresponding to the negative eigenvalue remains above it and tends to the origin. Any trajectory that starts at a point on the straight line corresponding to the positive eigenvalue remains below it and tends to the origin.