

## Difference equations – examples

**Example 6.** There is given the sequence 1, 1, 2, 3, 5, 8, 13, 21, ....

Determine its 100<sup>th</sup> member (Fibonacci sequence)

### SOLUTION

**Example 6.** The general member of the sequence has the form:  $u_{n+2} = u_{n+1} + u_n$ . We write the characteristic equation  $\rho(z) = z^2 - z - 1 = 0$  and find its roots:  $z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$ . Thus the general solution is obtained from the formula:

$$u_n = C_1 \left( \frac{1 - \sqrt{5}}{2} \right)^n + C_2 \left( \frac{1 + \sqrt{5}}{2} \right)^n.$$

By substitution with the initial conditions we calculate  $C_1 = \frac{1 - \sqrt{5}}{2\sqrt{5}}$ ,  $C_2 = \frac{1 + \sqrt{5}}{2\sqrt{5}}$ .

The solution of the problem is:  $u_{100} = \frac{1 - \sqrt{5}}{2\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{100} + \frac{1 + \sqrt{5}}{2\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{100}$ .

**Note:** By using the system *Mathematica* the exact value of the number in question is found in this way:

```
(* Example 6 Fibonacci numbers - difference equations*)
z = .
Solve[z^2 - z - 1 == 0, z] (*Solving the characteristic equation*)
t = z /. % (* Memorizing the roots in a list*)
z1 = t[[1]] (* Extraction of different roots*)
z2 = t[[2]]

{{z -> 1/2 (1 - Sqrt[5])}, {z -> 1/2 (1 + Sqrt[5])}}

{1/2 (1 - Sqrt[5]), 1/2 (1 + Sqrt[5])}

1/2 (1 - Sqrt[5])

1/2 (1 + Sqrt[5])

{{z -> 1/2 (1 - Sqrt[5])}, {z -> 1/2 (1 + Sqrt[5])}}
```

```
c1=.; c2=.; (* Solving the system about c1, c2*)
```

```
Solve[{c1 + c2 == 1,  
      c1 * z1 + c2 * z2 == 1}, {c1, c2}]
```

$$\left\{ \left\{ c1 \rightarrow -\frac{1}{10} \left( -5 + \sqrt{5} \right), c2 \rightarrow \frac{1 + \sqrt{5}}{2\sqrt{5}} \right\} \right\}$$

$$c1 = -\frac{1}{10} \left( -5 + \sqrt{5} \right); c2 = \frac{1 + \sqrt{5}}{2\sqrt{5}};$$

```
u100 = c1 * z1100 + c2 * z2100
```

```
Simplify[%]
```

```
% // N
```

$$\frac{\left(1 - \sqrt{5}\right)^{100} \left(5 - \sqrt{5}\right)}{12676506002282294014967032053760} + \frac{\left(1 + \sqrt{5}\right)^{101}}{2535301200456458802993406410752\sqrt{5}}$$

```
573147844013817084101
```

```
5.73148 × 1020
```

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