Difference equations – examples

Example 4. Find the solution of the difference equation.

\[
\begin{align*}
  a) & \quad u_{n+1} = \frac{1}{2} u_n + 1, \quad u_0 = 1 \\
  b) & \quad u_{n+2} - 3u_{n+1} + 2u_n = -1, \quad u_0 = 1, \ u_1 = 2 \\
  c) & \quad u_{n+1} - \frac{11}{6}u_n + \frac{1}{6}u_{n-2} = 0, \quad u_0 = 0, \ u_1 = 1, \ u_2 = 2 \\
  d) & \quad 6u_{n+4} - 5u_{n+2} + u_n = 0, \quad u_0 = 0, \ u_1 = \frac{1}{\sqrt{2}}, \ u_2 = \frac{1}{2}, \ u_3 = \frac{1}{2\sqrt{2}}
\end{align*}
\]

SOLUTIONS

We will use the following notations: \(u_n\) - general solution, \(\varphi_n\) - general solution of the homogeneous equation, \(v^*\) - particular solution of the non-homogeneous equation.

Example 4. a) This is a nonlinear homogeneous equation of the first order. We represent it in a standard form

\[u_{n+1} - \frac{1}{2}u_n = 1.
\]

Its corresponding homogeneous equation is: \(u_{n+1} - \frac{1}{2}u_n = 0\). Firstly we solve this homogeneous equation. We write down its characteristic equation: \(z - \frac{1}{2} = 0\). Obviously it has a root \(z = \frac{1}{2}\). Then the general solution of the homogeneous equation has the form \(\varphi_n = C_1 \left(\frac{1}{2}\right)^n\).

Then we need to find at least one particular solution of the given non-homogeneous equation. As its right hand side is a constant, we are looking for a particular solution in the form: \(v^* = d\), where \(d\) is a constant. We have \(d - \frac{1}{2}d = 1\), from where \(d = 2\), i.e. one particular solution is \(v^* = 2\). By properties 3 and 4 the general solution of the equation is a sum of the solutions of the homogeneous equation plus a particular solution, or the general solution of our equation is:

\[u_n = C_1 \left(\frac{1}{2}\right)^n + 2.
\]

For \(n = 0\) from the given initial condition \(u_0 = 1\), by substituting it in the general solution we obtain \(C_1 = -1\). The particular solution of the problem for the assigned initial condition that we were looking for is: \(u_n = 2 - \left(\frac{1}{2}\right)^n\), which is the answer to the given problem.
b) The equation is linearly non-homogeneous of the second order. As in the previous example, firstly we are looking for the general solution of the homogeneous equation.

\[ u_{n+2} - 3u_{n+1} + 2u_n = 0. \]

The characteristic equation \( z^2 - 3z + 2 = 0 \) has simple roots \( z_1 = 1, \ z_2 = 2 \). Therefore the general solution of the homogeneous equation is \( \overline{v}_n = C_1 + C_2 2^n \). Now we are looking for at least one particular solution of the non-homogeneous equation. As its right hand side is \(-1\), i.e. a constant, first of all we try a particular solution in the form \( v^* = d \). By substitution we obtain \( 0 = 0 \). Next we undertake the procedure in the form \( v^* = d \cdot n \). This time we get the equation \( d(n + 2) - 3d(n + 1) + 2dn = -1 \). After equating in front of the same monomials we find \( d = 1 \), i.e. we have a particular solution \( v^* = n \). In accordance with properties 3\(^0\) and 4\(^0\), the general solution of the non-homogeneous equation is represented in the form: \( u_n = \overline{v}_n + v^* \), i.e.

\[ u_n = C_1 + C_2 2^n + n. \]

By substitution under the assigned initial conditions we obtain the following system for \( C_1, C_2 \):

\[
\begin{align*}
C_1 + C_2 &= 1 \\
C_1 + 2C_2 &= 1.
\end{align*}
\]

Its solutions are \( C_1 = 1, \ C_2 = 0 \). Therefore the solution of the problem is \( u_n = 1 + n \).

c) The equation is linearly homogeneous of the third order. Its characteristic equation is

\[ z^3 - \frac{11}{6} z^2 + z - \frac{1}{6} = 0. \]

By using Horner’s method, by expansion, through the Mathematica system or in another way we find its roots \( z_1 = 1, \ z_2 = \frac{1}{2}, \ z_3 = \frac{1}{3} \), which are simple. Therefore the general solution of the given equation has the form:

\[ \overline{v}_n = C_1 + C_2 \left( \frac{1}{2} \right)^n + C_3 \left( \frac{1}{3} \right)^n. \]

By substitution of the assigned initial conditions for \( n = 0, 1, 2 \) we get the following system for determining the constants \( C_1, C_2, C_3 \): 

\[
\begin{align*}
C_1 + C_2 + C_3 &= 0 \\
C_1 + \frac{1}{2} C_2 + \frac{1}{3} C_3 &= 1 \\
C_1 + \frac{1}{4} C_2 + \frac{1}{9} C_3 &= 2.
\end{align*}
\]
Its solution is: \( C_1 = \frac{7}{2}, \ C_2 = -8, \ C_3 = \frac{9}{2}. \)

Answer: \( u_n = -8 \left( \frac{1}{2} \right)^n + \frac{9}{2} \left( \frac{1}{3} \right)^n + \frac{7}{2}. \)

**Note.** We submit the corresponding calculations by means of the system *Mathematica*:

\[
\begin{align*}
\text{(\dagger Example 4c - difference equations\(\star\))} \\
\text{\(z\).} \\
\text{\(\text{Solve}\left[z^3 - \frac{11}{6} z^2 + z - \frac{1}{6} = 0, \ z\right]\)} \\
\text{\(\{\{z \to \frac{1}{3}\}, \{z \to \frac{1}{2}\}, \{z \to 1\}\}\)} \\
\text{\(\text{Clear}[c_1, c_2, c_3]\)} \\
\text{\(\text{Solve}\left[[c_1 + c_2 + c_3 = 0, \ c_1 + \frac{1}{2} c_2 + \frac{1}{3} c_3 = 1, \ c_1 + \frac{1}{4} c_2 + \frac{1}{9} c_3 = 2\right]\), \{c_1, c_2, c_3\}\)} \\
\text{\(\{\{c_1 \to \frac{7}{2}, \ c_2 \to -8, \ c_3 \to \frac{9}{2}\}\}\)}
\end{align*}
\]

(4) The equation is homogeneous. Its characteristic equation is the biquadratic equation \( 6z^4 - 5z^2 + 1 = 0 \), which has four simple roots \( z_1 = \frac{1}{\sqrt{2}}, \ z_2 = -\frac{1}{\sqrt{2}}, \ z_3 = \frac{1}{\sqrt{3}}, \ z_4 = -\frac{1}{\sqrt{3}} \). Then the general solution of the difference equation has the form:

\[
u_n = C_1 \left( \frac{1}{\sqrt{2}} \right)^n + C_2 \left( -\frac{1}{\sqrt{2}} \right)^n + C_3 \left( \frac{1}{\sqrt{3}} \right)^n + C_4 \left( -\frac{1}{\sqrt{3}} \right)^n.
\]

Hence for \( n = 0,1,2,3 \) and from the given initial conditions we obtain the system with respect of the constants \( C_1, C_2, C_3, C_4 \):

\[
\begin{align*}
C_1 + C_2 + C_3 + C_4 &= 0 \\
\frac{1}{\sqrt{2}} C_1 - \frac{1}{\sqrt{2}} C_2 + \frac{1}{\sqrt{3}} C_3 - \frac{1}{\sqrt{3}} C_4 &= \frac{1}{\sqrt{2}} \\
\frac{1}{2} C_1 + \frac{1}{2} C_2 + \frac{1}{3} C_3 + \frac{1}{3} C_4 &= \frac{1}{2} \\
\frac{1}{2\sqrt{2}} C_1 - \frac{1}{2\sqrt{2}} C_2 + \frac{1}{3\sqrt{3}} C_3 - \frac{1}{3\sqrt{3}} C_4 &= \frac{1}{2\sqrt{2}}.
\end{align*}
\]

Its solutions are: \( C_1 = 2, \ C_2 = 1, \ C_3 = -\frac{3}{2}, \ C_4 = -\frac{3}{2}. \)
Answer: \[ u_n = 2 \left( \frac{1}{\sqrt{2}} \right)^n + \left( -\frac{1}{\sqrt{2}} \right)^n - \frac{3}{2} \left( \frac{1}{\sqrt{3}} \right)^n - \frac{3}{2} \left( -\frac{1}{\sqrt{3}} \right)^n. \]

**Note.** We submit the corresponding calculations for solving the system by means of *Mathematica*:

\[
\begin{align*}
\text{Solve}\left[ \{ \frac{1}{\sqrt{2}} c_1 - \frac{1}{\sqrt{2}} c_2 + \frac{1}{\sqrt{3}} c_3 - \frac{1}{\sqrt{3}} c_4 = \frac{1}{\sqrt{2}}, \\
\frac{1}{2} c_1 + \frac{1}{2} c_2 + \frac{1}{3} c_3 + \frac{1}{3} c_4 = \frac{1}{2}, \\
\frac{1}{2 \sqrt{2}} c_1 - \frac{1}{2 \sqrt{2}} c_2 + \frac{1}{3 \sqrt{3}} c_3 - \frac{1}{3 \sqrt{3}} c_4 = \frac{1}{2 \sqrt{2}} \}, \{ c_1, c_2, c_3, c_4 \} \right] \\
\{ \{ c_1 \to 2, c_2 \to 1, c_3 \to -\frac{3}{2}, c_4 \to -\frac{3}{2} \} \}
\end{align*}
\]

Author: Snezhana Gocheva-Ilieva
Plovdiv University
snow@uni-plovdiv.bg