Difference equations – examples

Example 4. Find the solution of the difference equation.

$$a)\ u_{n+1} = \frac{1}{2}u_n + 1, \qquad u_0 = 1$$

$$b)\ u_{n+2} - 3u_{n+1} + 2u_n = -1, \qquad u_0 = 1,\ u_1 = 2$$

$$c)\ u_{n+1} - \frac{11}{6}u_n + u_{n-1} - \frac{1}{6}u_{n-2} = 0, \quad u_0 = 0,\ u_1 = 1,\ u_2 = 2$$

$$d)\ 6u_{n+4} - 5u_{n+2} + u_n = 0\ , \qquad u_0 = 0,\ u_1 = \frac{1}{\sqrt{2}},\ u_2 = \frac{1}{2},\ u_3 = \frac{1}{2\sqrt{2}}$$

SOLUTIONS

We will use the following notations: u_n - general solution, $\overline{v}_n$ - general solution of the homogeneous equation, $v*$ - particular solution of the non-homogeneous equation.

Example 4. a) This is a nonlinear homogeneous equation of the first order. We represent it in a standard form

$$u_{n+1} - \frac{1}{2}u_n = 1.$$

Its corresponding homogeneous equation is: $u_{n+1} - \frac{1}{2}u_n = 0$. Firstly we solve this homogeneous equation. We write down its characteristic equation: $z - \frac{1}{2} = 0$. Obviously it has a root $z = \frac{1}{2}$. Then the general solution of the homogeneous equation has the form $\overline{v}_n = C_1\left(\frac{1}{2}\right)^n$.

Then we need to find at least one particular solution of the given non-homogeneous equation. As its right hand side is a constant, we are looking for a particular solution in the form: $v^* = d$, where d is a constant. We have $d - \frac{1}{2}d = 1$, from where $d = 2$, i.e. one particular solution is $v^* = 2$. By properties 3^0 and 4^0 the general solution of the equation is a sum of the solutions of the homogeneous equation plus a particular solution, or the general solution of our equation is:

$$u_n = C_1\left(\frac{1}{2}\right)^n + 2.$$

For $n = 0$ from the given initial condition $u_0 = 1$, by substituting it in the general solution we obtain $C_1 = -1$. The particular solution of the problem for the assigned initial condition that we were looking for is: $u_n = 2 - \left(\frac{1}{2}\right)^n$, which is the answer to the given problem.

b) The equation is linearly non-homogeneous of the second order. As in the previous example, firstly we are looking for the general solution of the homogeneous equation.

$$u_{n+2} - 3u_{n+1} + 2u_n = 0 .$$

The characteristic equation $z^2 - 3z + 2 = 0$ has simple roots $z_1 = 1,\ z_2 = 2$. Therefore the general solution of the homogeneous equation is $\overline{v}_n = C_1 + C_2 2^n$. Now we are looking for at least one particular solution of the non-homogeneous equation. As its right hand side is -1, i.e. a constant, first of all we try a particular solution in the form $v^* = d$. By substitution we obtain $0 = 0$. Next we undertake the procedure in the form $v^* = d.n$. This time we get the equation $d(n+2) - 3d(n+1) + 2dn = -1$. After equating in front of the same monomials we find $d = 1$, i.e. we have a particular solution $v^* = n$. In accordance with properties 3^0 and 4^0, the general solution of the non-homogeneous equation is represented in the form: $u_n = \overline{v}_n + v^*$, i.e.

$$u_n = C_1 + C_2 2^n + n .$$

By substitution under the assigned initial conditions we obtain the following system for $C_1,\ C_2$:

$$\left| \begin{array}{l} C_1 + C_2 = 1 \\ C_1 + 2C_2 = 1 \end{array} \right. .$$

Its solutions are $C_1 = 1,\ C_2 = 0$. Therefore the solution of the problem is $u_n = 1 + n$.

c) The equation is linearly homogeneous of the third order. Its characteristic equation is

$$z^3 - \frac{11}{6} z^2 + z - \frac{1}{6} = 0 .$$

By using Horner's method, by expansion, through the Mathematica system or in another way we find its roots $z_1 = 1,\ z_2 = \frac{1}{2},\ z_3 = \frac{1}{3}$, which are simple. Therefore the general solution of the given equation has the form:

$$\overline{v}_n = C_1 + C_2 \left(\frac{1}{2}\right)^n + C_3 \left(\frac{1}{3}\right)^n .$$

By substitution of the assigned initial conditions for $n = 0,1,2$ we get the following system for determining the constants $C_1,\ C_2,\ C_3$:

$$\left| \begin{array}{l} C_1 + C_2 + C_3 = 0 \\ C_1 + \frac{1}{2} C_2 + \frac{1}{3} C_3 = 1 \\ C_1 + \frac{1}{4} C_2 + \frac{1}{9} C_3 = 2 \end{array} \right. .$$

Its solution is: $C_1 = \frac{7}{2}$, $C_2 = -8$, $C_3 = \frac{9}{2}$.

Answer: $u_n = -8\left(\frac{1}{2}\right)^n + \frac{9}{2}\left(\frac{1}{3}\right)^n + \frac{7}{2}$.

Note. We submit the corresponding calculations by means of the system *Mathematica*:

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(* Example 4c - difference equations*)
z =.
Solve[z^3 - 11/6 z^2 + z - 1/6 == 0, z]
{{z -> 1/3}, {z -> 1/2}, {z -> 1}}

Clear[c1, c2, c3]
Solve[{c1 + c2 + c3 == 0, c1 + 1/2 c2 + 1/3 c3 == 1, c1 + 1/4 c2 + 1/9 c3 == 2
 }, {c1, c2, c3}]

{{c1 -> 7/2, c2 -> -8, c3 -> 9/2}}
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d) The equation is homogeneous. Its characteristic equation is the biquadratic equation $6z^4 - 5z^2 + 1 = 0$, which has four simple roots $z_1 = \frac{1}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}}$, $z_3 = \frac{1}{\sqrt{3}}$, $z_4 = -\frac{1}{\sqrt{3}}$. Then the general solution of the difference equation has the form:

$$u_n = C_1\left(\frac{1}{\sqrt{2}}\right)^n + C_2\left(-\frac{1}{\sqrt{2}}\right)^n + C_3\left(\frac{1}{\sqrt{3}}\right)^n + C_4\left(-\frac{1}{\sqrt{3}}\right)^n.$$

Hence for $n = 0,1,2,3$ and from the given initial conditions we obtain the system with respect of the constants C_1, C_2, C_3, C_4:

$$\left|\begin{array}{l} C_1 + C_2 + C_3 + C_4 = 0 \\ \frac{1}{\sqrt{2}}C_1 - \frac{1}{\sqrt{2}}C_2 + \frac{1}{\sqrt{3}}C_3 - \frac{1}{\sqrt{3}}C_4 = \frac{1}{\sqrt{2}} \\ \frac{1}{2}C_1 + \frac{1}{2}C_2 + \frac{1}{3}C_3 + \frac{1}{3}C_4 = \frac{1}{2} \\ \frac{1}{2\sqrt{2}}C_1 - \frac{1}{2\sqrt{2}}C_2 + \frac{1}{3\sqrt{3}}C_3 - \frac{1}{3\sqrt{3}}C_4 = \frac{1}{2\sqrt{2}} \end{array}\right. .$$

Its solutions are: $C_1 = 2$, $C_2 = 1$, $C_3 = -\frac{3}{2}$, $C_4 = -\frac{3}{2}$.

Answer: $u_n = 2\left(\frac{1}{\sqrt{2}}\right)^n + \left(-\frac{1}{\sqrt{2}}\right)^n - \frac{3}{2}\left(\frac{1}{\sqrt{3}}\right)^n - \frac{3}{2}\left(-\frac{1}{\sqrt{3}}\right)^n$.

Note. We submit the corresponding calculations for solving the system by means of *Mathematica*:

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(* Example 4d - difference equations*)
Solve[{c1 + c2 + c3 + c4 == 0,
   1/Sqrt[2] c1 - 1/Sqrt[2] c2 + 1/Sqrt[3] c3 - 1/Sqrt[3] c4 == 1/Sqrt[2],
   1/2 c1 + 1/2 c2 + 1/3 c3 + 1/3 c4 == 1/2,
   1/(2 Sqrt[2]) c1 - 1/(2 Sqrt[2]) c2 + 1/(3 Sqrt[3]) c3 - 1/(3 Sqrt[3]) c4 == 1/(2 Sqrt[2])}, {c1, c2, c3, c4}]
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$$\left\{\left\{c1 \to 2,\ c2 \to 1,\ c3 \to -\frac{3}{2},\ c4 \to -\frac{3}{2}\right\}\right\}$$

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