Difference equations – examples

Example 3. Prove that the sequences

$$v^{(1)} = \{1\}_{n=0}^{\infty}, \; v^{(2)} = \{n\}_{n=0}^{\infty}, \; v^{(3)} = \left\{n^2\right\}_{n=0}^{\infty}$$

are linearly independent solutions of the difference equation:

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = 0$$

SOLUTION

Problem 3. It is easy to check by substitution that $v^{(1)} = \{1\}_{n=0}^{\infty}$ is a solution of the given equation. Indeed:

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = 1^{n+3} - 3.1^{n+2} + 3.1^{n+1} - 1^n = 0 .$$

By analogy for $v^{(2)} = \{n\}_{n=0}^{\infty}$ we have :

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = n + 3 - 3(n+2) + 3(n+1) - n = 0 .$$

For the third sequence $v^{(3)} = \left\{n^2\right\}_{n=0}^{\infty}$:

$$u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n = (n+3)^2 - 3(n+2)^2 + 3(n+1)^2 - n^2 = 0 .$$

Thus we have proved that the three sequences satisfy the equation.

Their linear independence follows from the fact that the determinant from the first 3 members of the three sequences is different from zero. So we calculate

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 \neq 0 \quad \text{(see property } 2^0\text{)} .$$