Difference equations – examples

Example 1. Find the general solution of the difference equations:

$$a)\ \ u_{n+1} = \frac{1}{2}u_n$$
$$b)\ \ u_{n+2} = u_n$$
$$c)\ \ 6u_{n+2} - u_{n+1} - 2u_n = 0$$
$$d)\ \ u_{n+1} + u_n + u_{n-1} = 0$$
$$e)\ \ u_{n+1} = \frac{27}{10}u_n + \frac{1}{2}u_{n-1} + \frac{1}{5}u_{n-2}$$
$$f)\ \ u_{n+1} = u_n - \frac{1}{4}u_{n-1}$$
$$g)\ \ u_{n+1} = \frac{1}{6}u_n + \frac{1}{3}u_{n-1} + \frac{1}{2}$$
$$h)\ \ u_{n+1} - 2u_n + u_{n-1} = 2$$
$$i)\ \ u_{n+4} - 6u_{n+2} + 8u_n = 2^n$$
$$j)\ \ u_{n+3} + u_{n+2} - \frac{7}{4}u_{n+1} + \frac{1}{2}u_n = \frac{1}{2^n} + \frac{1}{3^n}$$

SOLUTIONS, ANSWERS

Example 1. a) The characteristic equation $\rho(z) = z - \frac{1}{2} = 0$ has a real root $z_1 = \frac{1}{2}$, $|z_1| < 1$, therefore the general solution of the difference equation is: $u_n = C\left(\frac{1}{2}\right)^n$.

b) As the corresponding characteristic equation $\rho(z) = z^2 - 1 = 0$ has two different real roots $z_1 = -1$, $z_2 = 1$, the general solution of the difference equation is written down in the following form: $u_n = C_1(-1)^n + C_2$.

c) Solution: $u_n = C_1\left(-\frac{1}{2}\right)^n + C_2\left(\frac{2}{3}\right)^n$.

d) The characteristic equation $\rho(z) = z^2 + z + 1 = 0$ has two complexly conjugated roots $z_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, from where we calculate that $r = 1$, $\theta = \frac{\pi}{3}$. The general solution of the difference equation will be $u_n = C_1 \cos\left(n\frac{\pi}{3}\right) + C_2 \sin\left(n\frac{\pi}{3}\right)$.

e) Answer: $u_n = C_1\left(\frac{1}{5}\right)^n + C_2\left(\frac{1}{2}\right)^n + C_3 2^n$.

f) For the characteristic equation $\rho(z) = z^2 - z + \frac{1}{4} = 0$ we obtain two identical real roots $z_{1,2} = \frac{1}{2}$, and for the general solution of the difference equation we get respectively:

$$u_n = C_1\left(\frac{1}{2}\right)^n + C_2 n\left(\frac{1}{2}\right)^n.$$

g) The roots of the characteristic equation are $z_1 = \frac{1}{2}$, $z_2 = \frac{2}{3}$, and the general solution of the homogeneous difference equation is: $\bar{v}_n = C_1\left(-\frac{1}{2}\right)^n + C_2\left(\frac{2}{3}\right)^n$. We are looking for a particular solution in the form $v^* = \beta$. By substitution in the given equation we find that $\beta = 1$. Finally, for the general solution of the non-homogeneous equation we get:

$$u_n = \bar{v}_n + v^* = C_1\left(-\frac{1}{2}\right)^n + C_2\left(\frac{2}{3}\right)^n + 1.$$

h) In this case the characteristic equation has a double real root $z_{1,2} = 1$. We are looking for the particular solution of the non-homogeneous equation in the form $v^* = \beta n^2$. Answer: $u_n = C_1 + C_2 n + n^2$.

i) Answer: $u_n = C_1(-2)^n + C_2 2^n + C_3(-\sqrt{2})^n + C_4(\sqrt{2})^n + \frac{1}{16} n 2^n$.

j) Answer: $u_n = C_1\left(\frac{1}{2}\right)^n + C_2 n\left(\frac{1}{2}\right)^n + C_3(-2)^n + \frac{4}{5} n^2\left(\frac{1}{2}\right)^n + \frac{108}{7}\left(\frac{1}{3}\right)^n$.

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