9.3. Bellman's optimality principle

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This principle was formulated by one of the originators of dynamic optimization - the American mathematician Richard Bellman. It states:

“Regardless of the decisions taken to enter a particular state in a particular stage, the remaining decisions made for leaving that stage must constitute an optimal policy.”

Consequence: **If we have entered the final state of an optimal policy we can trace it back.**

<u>Example 6.</u> With the help of Bellman’s optimality principle find the policy for minimal profit shown in fig. 3. Here the numbers of the nodes have been placed in rectangles and the length of the arcs (the prices) are the numbers in bold. The decisions taken are to be entered in the circles.

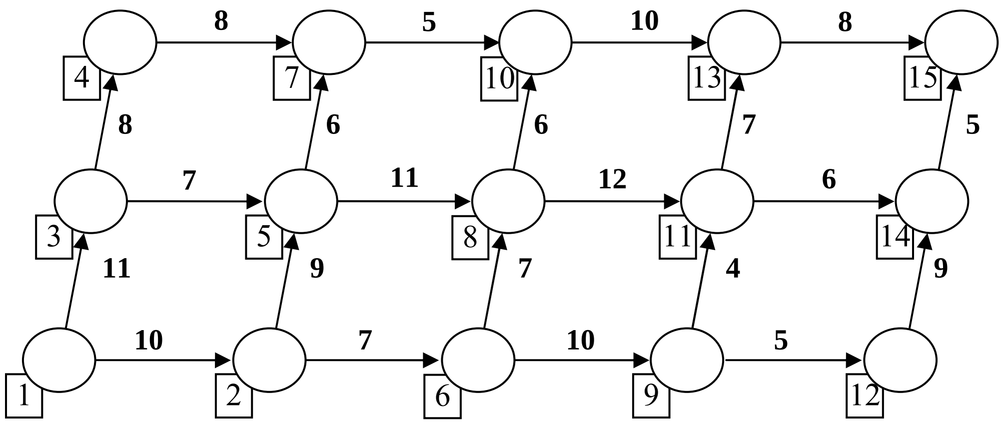
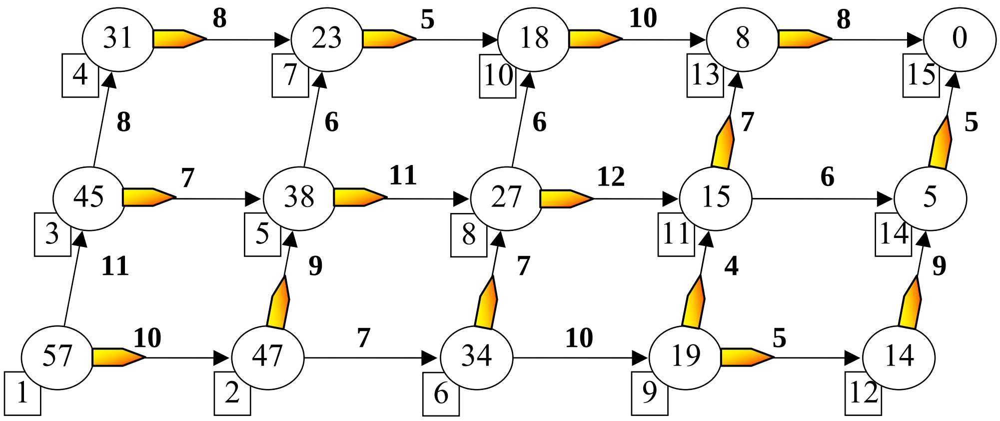


Fig. 3. Oriented weight graph for example 6.

Solution: It has been shown in fig. 4. According to Bellman's Optimality Principle we start from the end node $\boxed{15}$. In the circle we write a price of 0. We can reach this state only from nodes $\boxed{13}$ or $\boxed{14}$. If we are at node $\boxed{13}$ the only transition to $\boxed{15}$ has a price of 8, likewise the price of transition from $\boxed{14}$ to $\boxed{15}$ is 5 The transitions chosen are marked with a stylized marker . The next stage is filling in the decisions for states, $\boxed{11}$ and $\boxed{12}$. From $\boxed{10}$ there is a single transition to $\boxed{13}$ with a price of 10 + 8 = 18. From state $\boxed{11}$ there are to possibilities - through $\boxed{13}$, which costs 7 + 8 = 15 and $\boxed{14}$, which costs 6 + 5 = 11. Because we are looking for the maximal profit we choose the transition to $\boxed{13}$. From state $\boxed{12}$ we choose the only transition to $\boxed{14}$ with a total price of 14. From node $\boxed{9}$ there are to possible transitions which have identical prices, so both are acceptable. At node $\boxed{8}$ we choose a transition to $\boxed{11}$ the price of which is 27 > 24 and so on. We continue with this procedure back to state $\boxed{1}$ with a price of 57, which is the sought maximal price. The optimal policy which results in this maximal profit is achieved following stylized arrow: <1, 2, 5, 8, 11, 13, 15>. Profit check: 10 + 9 + 11 + 12 + 7 + 8 =57.



Ф Fig. 4. Solution to example 6 in the case of a maximum. Stylized arrows show possible transitions.

In the next fig. 5 we find the solution which results in a minimal price:

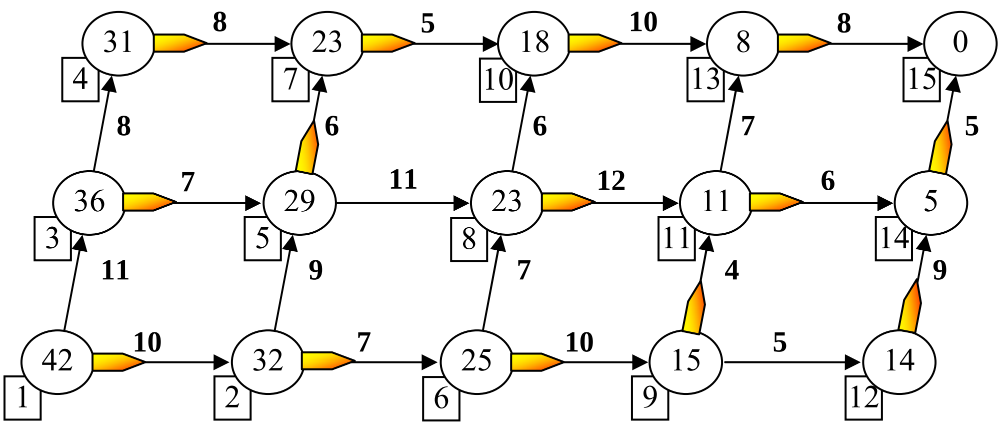
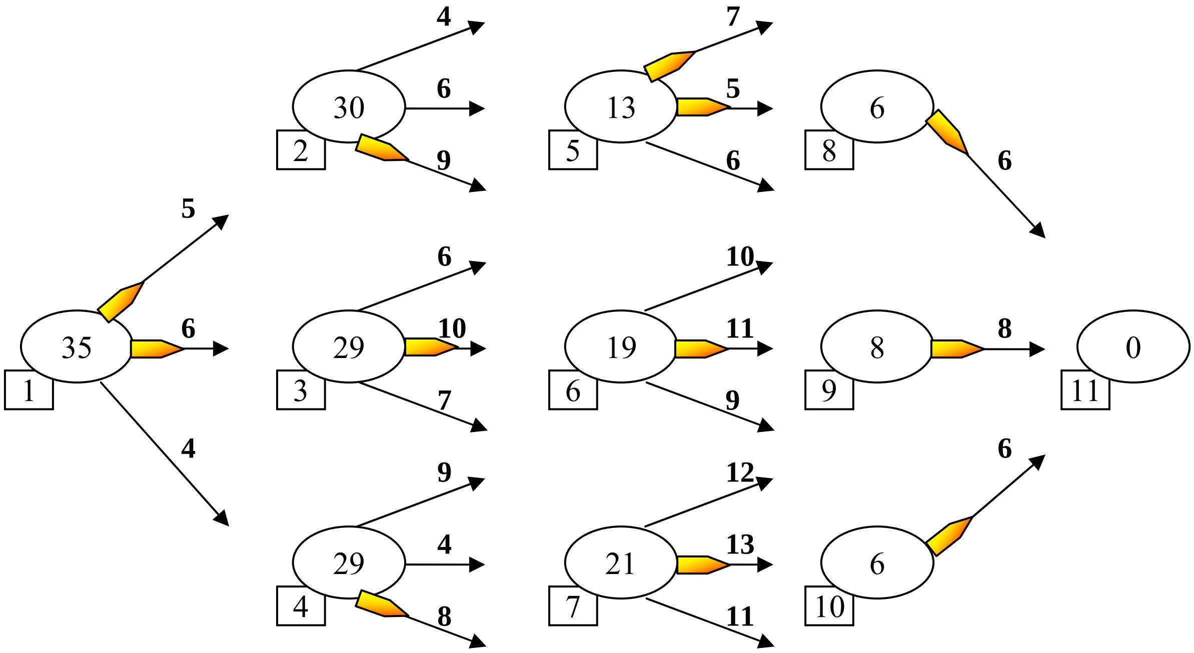


Fig. 5. Solution to example 6 in case of a minimum.

Answer: Minimum = 42, using strategy: <1, 2, 6, 9, 11, 14, 15>

<u>Example 7.</u> On the next fig. 6 is used Bellman's principle for finding the maximal profit for a process, which involves taking three decisions at every stage.



Преходи 4	Преходи 3	Преходи 2	Преходи 1
5+30=$\underline{35}$	4+13=17 6+19=25 9+21=$\underline{30}$	7+6=$\underline{13}$ 5+8=$\underline{13}$ 6+6=12	6+0=$\underline{6}$
6+29=$\underline{35}$	6+13=19 10+19=$\underline{29}$ 7+21=28	10+6=16 11+8=$\underline{19}$ 9+6=15	8+0=$\underline{8}$
4+29=33	9+13=22 4+19=23 8+21=$\underline{29}$	12+6=18 13+8=$\underline{21}$ 11+6=17	6+0=$\underline{6}$

Fig. 6. Results for example 7. The maximal transitions are underlined and shown using a stylized arrow.

The solution of the problem starts with the last node $\boxed{11}$, where we put the price of 0. Reaching this point is possible from nodes $\boxed{8}$, $\boxed{9}$ or $\boxed{10}$ in one way only, with transition prices respectively 6, 8, 6. If we are in state $\boxed{5}$, we can go through $\boxed{8}$, $\boxed{9}$ or $\boxed{11}$. We find the respective maximal price from the three transitions, i.e. max {7+6, 5+8, 6+6}= 13. It is reached for two possibilities - through $\boxed{8}$ and $\boxed{9}$, where we place the stylized arrows . Likewise if we are at node $\boxed{6}$ the maximal of the three solutions is max {10+6, 11+8, 9+6}= 19. We continue in a similar manner back to node $\boxed{1}$. Maximal profit is 35, which is achievable using two optimal policies, from node $\boxed{1}$ in the direction of stylized arrows to the end. They are: <1, 2, 7, 9, 11> and <1, 3, 6, 9, 11>. All intermediate calculations have been filled in the table in fig.6. For additional exercise find the optimal policy for minimal profit.

Notes on the implementation of Bellman's principle

The first example demonstrates the implementation of Bellman's principle with N number of stages, with no more than two decisions taken at every stage. This requires the carrying out of $R = O(2^n)$ arithmetical operations. In the example that follows for every stage 3 decisions were taken, i.e. for an *n* number of stages there will be needed $O(3^n)$ calculations. In the general case of processes with *n* stages and taking an *m* number of decisions at every stage the requirement will be for the implementation of $R = O(m^n)$ arithmetical operations. Despite the simplicity of the method the number of arithmetical operations may become too great and even impossible to implement. For example with *m*=5 *n*=20 we get $R = O(5^{20})$ = 95367431640625. If a computer does the calculations with a speed of 1,000,000 arithmetical operations a second, it will take approximately 95367431 seconds, i.e. around 3 years of work around the clock! For this reason solving problems with greater parameters requires finding more effective methods suited to the type of problem.