Problems on iterative methods for solving systems of linear algebraic equations

1) Prove that for the system

$$\left|\begin{array}{rrrrcr} 10x_1 & -x_2 & +2x_3 & -3x_4 & = & 0 \\ x_1 & +10x_2 & -x_3 & -2x_4 & = & 5 \\ 3x_1 & +2x_2 & +x_3 & +20x_4 & = & 15 \\ 2x_1 & +3x_2 & +20x_3 & -x_4 & = & -10 \end{array}\right.$$

the method of simple iteration converges. If for the initial guess $x^{(0)}$ we choose the column of the right side vector, e.g. $x^{(0)} = (0,5,-10,15)^T$, how many iterations would be sufficient for calculating the solution with an accuracy of $\varepsilon = 10^{-4}$? Work in a second matrix and vector norms.

Instruction: You must rearrange the last two equations.

Answer: *k*=22 iterations.

2) Solve the following linear system using the method of simple iteration with an accuracy of $\varepsilon = 10^{-2}$.

$$\left|\begin{array}{rrrl} 2x_1 & -x_2 & +x_3 & =-3 \\ 3x_1 & +5x_2 & -2x_3 & =1 \\ x_1 & -4x_2 & +10x_3 & =0 \end{array}\right. .$$

Answer: (-1,212; 1,162; 0,586).

3) Solve the system $Ax = b$, where

$$A = \begin{pmatrix} 7,6 & 0,5 & 2,4 \\ 2,2 & 9,1 & 4,4 \\ -1,3 & 0,2 & 5,8 \end{pmatrix}, \quad b = \begin{pmatrix} 1,9 \\ 9,7 \\ -1,4 \end{pmatrix}.$$

using the Zeidel method with an accuracy of $\frac{1}{2}.10^{-2}$. The system must be transformed to a form that is suitable of iteration $x = Bx + c$, where $B = E - 0{,}1A$, $c = 0{,}1b$. Choose $x^{(0)} = c$.

Answer: If you are working correctly you should derive the following approximations:

$x^{(0)}$ = (0,19; 0,97; -0,14),

$x^{(1)}$ = (0,2207; 1,0703; -0,1915),

$x^{(2)}$ = (0,2354; 1,0988; -0,2118),

$x^{(3)}$ = (0,2424; 1,1088; -0,2196),

$x^{(4)}$ = (0,2425; 1,1124; -0,2226) .

4) The following system is given

$$\left| \begin{array}{ccccc} x_1 & & -0{,}25x_3 & -0{,}25x_4 & = 0{,}5 \\ & x_2 & -0{,}25x_3 & -0{,}25x_4 & = 0{,}5 \\ -0{,}25x_1 & -0{,}25x_2 & +x_3 & & = 0{,}5 \\ -0{,}25x_1 & -0{,}25x_2 & & +x_4 & = 0{,}5 \end{array} \right.$$

a) By starting from $x^{(0)} = (0,\ 0,\ 0,\ 0)^T$, do 4 iterations using the Jacobi method.

b) By using the same initial guess vector, do 4 iterations using the Zeidel method.

c) Which is the accurate solution of the system?

5) The system $Ax = b$ is given, where

$$A = \begin{pmatrix} 0{,}95 & -0{,}1 & 0{,}1 \\ 0{,}15 & 1{,}1 & -0{,}05 \\ -0{,}1 & 0{,}1 & 1{,}05 \end{pmatrix}, \quad b = \begin{pmatrix} 0{,}85 \\ 0{,}20 \\ -1{,}15 \end{pmatrix}.$$

a) Build an iteration process of the kind $x^{(k+1)} = Bx^{(k)} + b$, where $B = E - A$.

b) Prove that the iteration process converges.

c) Find the minimum number of iterations which guarantees an accuracy of 10^{-6} with $x^{(0)} = b$ in second norm.

d) Calculate $x^{(3)}$, by rounding off up to the third digit after the decimal point (included). Answer: The exact solution is (1, 0, -1).

6) A system of linear algebraic equations is given. Solve it using a suitably chosen method and an accuracy of your choice.

$$\left| \begin{array}{rrrl} 4x_1 & +0{,}24x_2 & -0{,}08x_3 & =8 \\ 0{,}09x_1 & +3x_2 & -0{,}15x_3 & =9 \\ 0{,}04x_1 & -0{,}08x_2 & +4x_3 & =20 \end{array} \right. .$$

Answer: Tentatively at $x^{(0)} = (2,3,5)^T$ the third iteration gives

$x^{(3)} = (1{,}90923;\ 3{,}19495;\ 5{,}04485)$.

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