Problems on numerical methods for eigenvalues and eigenvectors of matrices

Find the characterstic polynomial ***P(λ)***, the eigevalues ***λ*** and the eigenvectors ***x*** of the matrices using the given initial vector c^0:

1) $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) $c^0 = (0,1,0)^T$, $P_3(\lambda) = ?$ by the Lanczos method;

b) $c^0 = (1,0,0)^T$, $P_2(\lambda)$ is the divisor of $P_3(\lambda) = ?$;

c) with biortogonalization method and $c^0 = b^0 = (0,0,1)^T$.

Answer: $P_3(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6$

$\lambda = 3,\ x = (1,1,1)^T$; $\lambda = -2,\ x = (11,1,-14)^T$; $\lambda = 1,\ x = (-1,1,1)^T$.

2) $A = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 4 & 5 \\ 3 & 2 & -1 \end{bmatrix}$

a) $c^0 = (1,0,0)^T$, by the Lanczos method,

b) $c^0 = b^0 = (1,0,0)^T$ when using biortogonalization method.

Answer: $P_3(\lambda) = \lambda^3 - 5\lambda^2 - 19\lambda + 89$; λ are found using a method for solving a non-linear equation: $\lambda_1 \approx -4{,}284$; $\lambda_2 = 3{,}761$; $\lambda_3 = 5{,}522$.

The eigenvectors can be found using a computer program.

3) $A = \begin{bmatrix} 16 & -24 & 18 \\ 3 & -2 & 0 \\ -9 & 18 & -17 \end{bmatrix}$,

a) $c^0 = (1,1,1)^T$. Find the divisor $P_2(\lambda)$;

b) $c^0 = (0,0,1)^T$, find $P_3(\lambda) = ?$

Answer: $P_3(\lambda) = \lambda^3 + 3\lambda^2 - 36\lambda + 32$,

$\lambda = 1,\ x = (2,2,1)^T$; $\lambda = 4,\ x = (2,1,0)^T$; $\lambda = -8,\ x = (-2,1,4)^T$.

4) $A = \begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}$, $c^0 = (0,1,0,0)^T$.

Answer: $P_3(\lambda) = \lambda^3 - 19\lambda^2 + 55\lambda + 75$ – divisor of the characteristic polynomial. $\lambda_{1,2} = 5$, $x = (-1,0,0,1)^T$; $\lambda_3 = -1$, $x = (1,-1,-1,1)^T$; $\lambda_3 = 15$, $x = (1,1,1,1)^T$.

5) $A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 3 & 5 \\ 5 & 5 & -1 \end{bmatrix}$, $c^0 = (1,0,0)^T$.

Answer: $P_3(\lambda) = \lambda^3 - 5\lambda^2 - 48\lambda + 108$;

$\lambda_1 = 2$, $x = (1,-1,0)^T$; $\lambda_2 = -6$, $x = (-1,-1,2)^T$; $\lambda_3 = 9$, $x = (1,1,1)^T$.

6) $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$,

a) $c^0 = (1,0,0,0)^T$,

b) $c^0 = (0,0,0,1)^T$.

Answer: $P_3(\lambda) = \lambda^4 - 6\lambda^3 + 8\lambda^2 + 2\lambda - 3$, the eigenvalues are found using approximation.

7) $A = \begin{bmatrix} -6 & -80 & 290 & 80 \\ -1 & -60 & 119 & 26 \\ -1 & -100 & 180 & 37 \\ 2 & 312 & -529 & -104 \end{bmatrix}.$

Answer: $P_4(\lambda) = \lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24;$

$\lambda = 1; 2; 3; 4; \quad \boldsymbol{x} = \begin{pmatrix} 130 \\ 21 \\ 23 \\ -51 \end{pmatrix}; \begin{pmatrix} 90 \\ 15 \\ 16 \\ -34 \end{pmatrix}; \begin{pmatrix} 70 \\ 11 \\ 11 \\ -21 \end{pmatrix}; \begin{pmatrix} 64 \\ 9 \\ 8 \\ -12 \end{pmatrix}.$

8) Calculate $\max\limits_{1 \le i \le n} |\lambda_i|$ for a given matrix A using the exponent method with accuracy of $\varepsilon = 10^{-2}; 10^{-3}; 10^{-4}$:

a) $A = \begin{bmatrix} 16 & -24 & -18 \\ 3 & -2 & 0 \\ -9 & 18 & -17 \end{bmatrix}$ Answer: $\lambda = -8$; $x = (-2, 1, 4)^T$

b) $A = \begin{bmatrix} 3 & 1 & 5 \\ 1 & 3 & 5 \\ 5 & 5 & -1 \end{bmatrix}$ Answer: $\lambda = 9$; $x = (0{,}97; 1; 0{,}99)^T \approx (1,1,1)^T$

c) $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ Answer: $\lambda = 8{,}3874$; $x = (0{,}8077; 0{,}772; 1)^T$

d) $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ Answer: $\lambda = 3{,}618$, $x = (0{,}37; -0{,}6; 0{,}6; -0{,}37)^T$.

9) Apply Jacobi's method to derive the eigenvalues and eigenvectors of the following matrix with an accuracy of $\varepsilon = 10^{-1}; 10^{-2}; 10^{-3}$:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Answer: $\lambda_1 \approx 3{,}4142$, $x_1 \sim (0{,}7071, -1, 0{,}7071)^T$;

$\lambda_2 \approx 0{,}5858$, $x_2 \sim (0{,}7071, 1, 0{,}7071)^T$; $\lambda_3 \approx 2{,}0000$, $x_3 \sim (1, 0, -1)^T$.

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