Bisection method for solving equations with a single unknown

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This method is also called dichotomy method.
Let the given equation be:

$$f(x) = 0, \tag{1}$$

where the function $f(x)$ is defined and continuous in the interval $[A, B]$.

Localizing a root requires finding a subinterval $[a, b]$ of interval $[A, B]$, so that in ends a, b the function $f(x)$ takes different signs i.e. $f(a).f(b) < 0$. Then according to a theorem from analysis the consequence is that in the interval $[a, b]$ there exists at least one root α of equation (1). As it is a root then $f(\alpha) = 0$.

The aim is to determine α with some accuracy $\varepsilon > 0$.

We mark $a_0 = a$, $b_0 = b$ and calculate the middle c_0 of the interval $[a_0, b_0]$ using the formula:

$$c_0 = \frac{a_0 + b_0}{2}$$

and consider the subintervals $[a_0, c_0]$ and $[c_0, b_0]$. To understand which one contains the root α we check the sign of the product of $f(a_0).f(c_0)$. If $f(a_0).f(c_0) < 0$, then $\alpha \in [a_0, c_0]$, otherwise $\alpha \in [c_0, b_0]$. The new interval of localization we mark with $[a_1, b_1]$. We calculate the middle $c_1 = \frac{a_1 + b_1}{2}$ and so on.

The resulting process creates a sequence of intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset \ldots \supset [a_n, b_n] \supset \ldots,$$

each of which contains α, i.e. $a_n \le \alpha \le b_n$, $n = 0, 1, \ldots$

Furthermore every integral is two times shorter than the previous and because of this

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \frac{b_{n-2} - a_{n-2}}{4} = \ldots = \frac{b - a}{2^n} \underset{n \to \infty}{\longrightarrow} 0.$$

From this, as

$$0 \le b_n - \alpha \le b_n - a_n = \frac{b - a}{2^n} \to 0$$

$$0 \le \alpha - a_n \le b_n - a_n = \frac{b - a}{2^n} \to 0$$

we get $\lim_{n\to\infty}(b_n - \alpha) = 0$ and $\lim_{n\to\infty}(\alpha - a_n) = 0$,

i.e. $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \alpha$.

The result is that both sequences $\{a_n\}$ and $\{b_n\}$ attend to the root α. To find it with a given accuracy ε, we can stop calculations when

$$|b_n - a_n| < \varepsilon .$$

The presented method of dividing into halves is one of the simplest and most reliable numerical methods for solving the equation $f(x) = 0$. In fact it can also be used to find the root when in $[A, B]$ there had been more than one root, for example 3, 5 or more roots. In the considered case we would calculate the leftmost of them. If we want to find the rightmost we need to chose the new interval of localization according to the sign in front of $f(c_n).f(b_n)$.

Example: In the chemical reaction

$$CO + \frac{1}{2}O_2 \rightleftarrows CO_2$$

the decomposition percentage $\underline{x}$ of 1 mol CO_2 can be found by the equation

$$\left(\frac{p}{k^2} - 1\right)x^3 + 3x - 2 = 0,$$

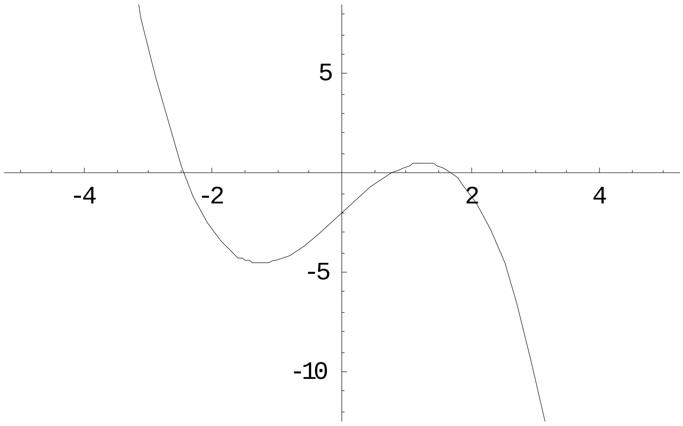
where *p* is the pressure of CO_2 in atmospheres and *k* is an equilibrium constant dependent on temperature. Calculate $\underline{x}$ when $p = 1$ atmospheres and $k = 1.648$ (which is equal to 2800 *K*).

Solution: In order to get to know the number of real roots and their approximate values we construct the graphic of the function

$$f(x) = \left(\frac{1}{1.648^2} - 1\right)x^3 + 3x - 2 .$$

in the interval [-5, 5] with the help of the *Mathematica* system. The code is:

```
f[x_] := (1/1.648^2 - 1) x^3 + 3 x - 2          (* Defining function f (x) *)
Plot[f[x], {x, -5, 5}]     (* Drawing the graph of the finction in [-5, 5]*)
```



It is clear from it that the equation $g(x)=0$ has three real roots $\alpha_1 \in [-3,-2]$, $\alpha_2 \in [0,1]$, $\alpha_3 \in [1,2]$. In these intervals the requirements of the bisection method are fulfilled. In table 1 are shown the results of the calculation of $\alpha_2 \in [0,1]$ to the accuracy of $\varepsilon = 0.001$. The intermediate results are calculated to the exactness of 0.0001.

n	a_n	b_n	$\|b_n - a_n\|$
0	0	1	1
1	0.5	1	0.5
2	0.75	1	0.25
3	0.75	0.875	0.125
4	0.75	0.8125	0.0625
5	0.75	0.7812	0.0316
6	0.75	0.7656	0.0156
7	0.7578	0.7656	0.0078
8	0.7578	0.7617	0.0039
9	0.7578	0.7598	0.0020
10	0.7578	0.7588	0.0010

Table. 1

The resulting approximate value of the root with accuracy of three digits is $\alpha_2 \approx 0.758$.

Here is an example code of the *Mathematica* system accompanied by comments:

```
(* Bisection method to find the real roots of the equation f(x)=0 in
the beginning of a given interval[a, b] with a given accuracy epsi
*)

a = 0.; b = 1.; epsi = 0.001;        (* Setting th interval and accuracy *)
Print[" Finding a real root with accuracy = ", epsi]; i = 0;     (* i is a counter variable *)
While[Abs[b - a] > epsi,
          (* Cycle to check if the current lenght of the sub-interval is  > epsi*)
    c = (a + b)/2.;              (* c - the middle point of the current new sub-interval*)
    If[f[a] * f[c] < 0, b = c, a = c];  i = i + 1;
      (* Chosing one of the half subintervals whte the root can be found,
         and renaming its boundary points again by  a, b*)
    Print[" i=", i, "   a=", a, "   b=", b, "   c=", c, "  f=", f[c]]
       (* Current value of  a,b,c  and corresponding f(c) *)
   ]
Print[" Last approximation to the real root=", c = (a + b)/2]
```

```
 Finding a root with accuracy =  0.001
 i= 1    a= 0.5     b= 1.    c= 0.5   f= -0.578975
 i= 2    a= 0.75    b= 1.    c= 0.75   f= -0.01654
 i= 3    a= 0.75    b= 0.875    c= 0.875   f= 0.201744
 i= 4    a= 0.75    b= 0.8125    c= 0.8125   f= 0.0986179
 i= 5    a= 0.75    b= 0.78125    c= 0.78125   f= 0.042485
 i= 6    a= 0.75    b= 0.765625    c= 0.765625   f= 0.0133268
 i= 7    a= 0.757813    b= 0.765625    c= 0.757813   f= -0.00151892
 i= 8    a= 0.757813    b= 0.761719    c= 0.761719   f= 0.00592597
 i= 9    a= 0.757813    b= 0.759766    c= 0.759766   f= 0.00220902
 i= 10   a= 0.757813    b= 0.758789    c= 0.758789   f= 0.00034642

 Last approximation = 0.758301
```