Examples for finite difference method for solving PDE of parabolic type

Problem 1. Solve the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in [0;0,6],\ t \in (0;0,01),$$

which, for $x \in [0;0,6],\ t \in [0;0,01]$ satisfies the following conditions:

1.1. $u(x,0) = 3x(1-x) + 0,12$

$u(0;t) = 2(t + 0,06)$

$u(0,6;t) = 0,84$

1.2. $u(x,0) = x(x+1)$

$u(0,t) = 0$

$u(0,6;t) = 2t + 0,96$

1.3. $u(x,0) = 3x(2-x)$

$u(0,t) = 0$

$u(0,6;t) = t + 2,52$

1.4. $u(x,0) = 2x(1-x) + 0,2$

$u(0,t) = 0,2$

$u(0,6;t) = t + 0,68$

1.5. $u(x,0) = 2x(x+0,2) + 0,4$

$u(0,t) = 2t + 0,4$

$u(0,6;t) = 1,36$

1.6. $u(x,0) = 0,3 + x(x+0,4)$

$u(0,t) = 0,3$

$u(0{,}6;t) = 6t + 0{,}9$

a) For the obvious stable difference scheme $h = 0{,}1$ and $\sigma = \frac{\tau}{h^2} = \frac{1}{6}$.

б) For the non-obvious scheme $h = 0{,}1$ and $\sigma = 1$.

Solution:

a) From (11) 1.1. $\frac{u_i^{j+1} - u_i^j}{\tau} - \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} = 0,\ i = \overline{1{,}5};\ j = \overline{0{,}5}$

$$u_i^{j+1} = \frac{1}{6}\left(u_{i+1}^j + 4u_i^j + u_{i-1}^j\right), \quad i = \overline{1{,}5};\ j = \overline{0{,}5}$$

$$u_i^0 = 3ih(1 - ih) + 0{,}12, \quad i = \overline{1{,}5}$$

$$u_0^j = 2(j\tau + 0{,}06)$$

$$u_6^j = 0{,}84 \qquad j = \overline{1{,}6} \quad h = 0{,}1 \quad \tau = 0{,}0017$$

For 1.2. $u_i^0 = ih(ih + 1)$

$u_0^j = 0$

$u_6^j = 2j\tau + 0{,}96$

For 1.3. $u_i^0 = 3ih(2 - ih)$

$u_0^j = 0$

$u_6^j = \tau j + 2{,}52$

For 1.4. $u_i^0 = 2ih(1 - ih) + 0{,}2$

$u_0^j = 0{,}2$

$u_6^j = \tau j + 0{,}68$

For 1.5. $u_i^0 = 2ih(ih + 0{,}2) + 0{,}4$

$u_0^j = 2\tau j + 0{,}4$

$u_6^j = 1{,}36$

For 1.6. $u_i^0 = 0{,}3 + ih(ih + 0{,}4)$

$u_0^j = 0{,}3$

$u_6^j = 2\tau j + 0{,}9$

For 1.2-1.6 the indices change as in 1.1:

$i = \overline{1{,}5} \quad j = \overline{1{,}6} \quad h = 0{,}1 \quad \tau = 0{,}0017$

b) In (12) for $\frac{\tau}{h^2} = 1$ we have:

$$\frac{u_i^{j+1}-u_i^j}{\tau}-\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2}=0 \qquad \Rightarrow$$

$u_{i-1}^{j+1}-3u_i^{j+1}+u_{i+1}^{j+1}=-u_i^j,$ for $\forall j$ $i=\overline{1,5}$ the system is solved by means of 3-diagonal matrices methods (for example the expulsion method). Conditions 1.1.–1.6 are added to the system above.

We will write down the system in a matrix form for 1.1. when $j=0$.

$$u_0^1-3u_1^1+u_2^1=-u_1^0$$
$$u_1^1-3u_2^1+u_3^1=-u_2^0$$
$$u_2^1-3u_3^1+u_4^1=-u_3^0$$
$$u_3^1-3u_4^1+u_5^1=-u_4^0$$
$$u_4^1-3u_5^1+u_6^1=-u_5^0$$

$$\begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \end{pmatrix}=\begin{pmatrix} -u_1^0-u_0^1 \\ -u_2^0 \\ -u_3^0 \\ -u_4^0 \\ -u_5^0-u_6^1 \end{pmatrix}$$

In the right-hand side $u_i^0=3ih(1-ih)$, $i=\overline{1,5}$ (initial conditions)

$$\left.\begin{aligned} u_0^1 &= 2(\tau+0{,}06) \\ u_6^1 &= 0{,}84 \end{aligned}\right\} \text{(boundary conditions)}$$

By analogy the same refers to $j=\overline{1,5}$, for which only the right-hand parts of the systems are changed.

(tables with the solutions)

Problem 2. Find the approximate solution of the equation

$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+2x+t$, by step $h=0,1$ on x, which satisfies the following initial and boundary conditions:

2.1. $u(x,0)=-\frac{1}{3}x^3$

$u(0,t)=\frac{1}{2}t^2,\ \ u(0,6;t)=\frac{1}{2}t^2-0,072$

2.2. $u(x,0;02)=-\frac{1}{3}x^3+0,0002,\ \ x\in[0,3;0,9]$

$u(0,3;t)=-0,009+\frac{1}{2}t^2,\ u(0,9;t)=-0,243+\frac{1}{2}t^2,\ t\in[0,02;0,03]$

2.3. $u(x;0,01)=-\frac{1}{3}x^3+0,00005\quad x\in[0,6;1,2]$

$u(0,6;t)=-0,072+\frac{1}{2}t^2,\ u(1,2;t)=-0,576+\frac{1}{2}t^2,\ t\in[0,01;0,02]$

2.4. $u(x;0,03)=-\frac{1}{3}x^3+0,00045,\ x\in[0,9;1,5]$

$u(0,9;t)=\frac{1}{2}t^2-0,243,\ u(1,5;t)=\frac{1}{2}t^2-1,125,\ t\in[0,03;0,04]$

a) When there is used a stable DS with $h=0,1$ and $\sigma=\frac{1}{2}$, and we work with 5 digits after the decimal point.

б) Write the non-obvious scheme by $h=0,1$ and $\sigma=1$ and work out the difference equations system for all the grid knots.

Solution:

a) By substituting the partial derivatives in the equation with difference relations from (7), (10), we obtain:

$$\frac{u_i^{j+1}}{\tau}=\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{h^2}+F_i^j$$

for $x \in [0;0,6],\ t \in [0;0,01],\ h = 0,1,\ \tau = 0,005,\ i = \overline{0,6},\ j = \overline{0,2}$

$F_i^j = 2x_i + t_j = 2ih + j\tau$.

We make an allowance regarding the layer solution $j+1$:

$$u_i^{j+1} = \frac{1}{2}\left(u_{i+1}^j + u_{i-1}^j\right) + 0,005F_i^j$$

From 2.1. we use $u_i^0 = -\frac{1}{3}(ih)^3,\ i = \overline{1,5}$

$$u_0^j = \frac{1}{2}(\tau j)^2,\ u_6^j = \frac{1}{2}(\tau j)^2 - 0,072,\ j = \overline{1,2}$$

We fill in the results in the table:

t \ x	0	0,1	0,2	0,3	0,4	0,5	0,6
0	0	-0,00033	-0,00267	-0,009	-0,02133	-0,04167	-0,072
0,005	0,00001	-0,00031	-0,00264	-0,00898	-0,02131	-0,04164	-0,07199
0,01	0,00005	-0,00027	-0,00259	-0,00893	-0,02126	-0,0416	-0,07195

For $u_{exact} = -\frac{1}{3}x^3 + \frac{1}{2}t^2$ the table:

t \ x	0	0,1	0,2	0,3	0,4	0,5	0,6
0,005	0,00001	-0,00032	-0,00265	-0,00899	-0,02132	-0,04165	
0,01		-0,00028	-0,00262	-0,00895	-0,02128	-0,04162	

By means of comparison we find that the error is of order $0,00004 \approx 0\left(h^2 + \tau\right) \sim C.0,015$, which is theoretically grounded.

b) The non-obvious scheme (12) for $\sigma = 1$

$$\frac{u_i^{j+1}-u_i^j}{\tau}=\frac{u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}}{h^2}+2x_i+t_j \quad \Leftrightarrow$$

$$u_{i-1}^{j+1}-3u_i^{j+1}+u_{i+1}^{j+1}=-u_i^j+\tau\left(2x_i+t_j\right)$$

under the conditions 2.2. $i=\overline{1,5}, \quad j=\overline{0,1}, \quad x_i=0,3+ih, \quad t_j=0,02+j\tau$.

By analogy with Problem 1 b) we write down:

$j=0$

$$\begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \end{pmatrix}=\begin{pmatrix} -u_1^0-u_0^1-\tau f_1^0 \\ -u_2^0-\tau f_2^0 \\ -u_3^0-\tau f_3^0 \\ -u_4^0-\tau f_4^0 \\ -u_5^0-u_6^1-\tau f_5^0 \end{pmatrix}$$

$$u_i^0=-\frac{1}{3}x_i^3+0,0002, \quad i=\overline{1,5}$$

$$u_0^1=-0,009+\frac{1}{2}0,02^2$$

$$u_6^1=-0,243+\frac{1}{2}0,02^2$$

$$f_i^0=2x_i+0,02$$

In order to solve the system we make use of the expulsion method, which is stable for these values of the 3-diagonal matrix elements.

Problem 3. (For individual work)

Solve the equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha\left(x^2-2t\right), \quad 0<x<1, \quad t\in\left(0;0,04\right), \quad \alpha=0,5k, \quad k=\overline{1,6},$$

which satisfies the following conditions:

$u(0,t)=0, \quad u(1,t)=\alpha t, \quad t \in [0;0,04]$

$u(x,0)=0, \quad 0 \le x \le 1$

a) using an obvious DS for $h=0,2 \quad \tau=0,02$

b) using a non-obvious DS for $h=0,2 \quad \tau=0,02$

There is known the exact solution $u=\alpha x^2 t$.

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