Finite difference method for solving PDE of parabolic type

We are considering the **Cauchy problem for the heat-conductivity equation:**

(3) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \varphi(x,t), \ -\infty < x < \infty, \ t > 0$$

(4) $$u(x,0) = \psi(x), \ x \in (-\infty, \infty)$$

We will presume that the problem (3)–(4) has a single solution u(x,t), which is continuous together with its derivatives

$$\frac{\partial^i u}{\partial t^i}, \ i = 1,2, \ \frac{\partial^k u}{\partial x^k}, \ k = \overline{1,4}.$$

We will write down (3)-(4) in form (1): $Lu = f$, i.e. we replace:

$$Lu \equiv \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2}, & x \in (-\infty,\infty), t > 0 \\ u(x,0), & x \in (-\infty,\infty) \end{cases}$$

$$f \equiv \begin{cases} \varphi(x,t), & x \in (-\infty,\infty), t > 0 \\ \psi(x), & t = 0 \end{cases}$$

Let $t \in [0,T]$, $T < \infty$. T.e. $D = \{-\infty < x < \infty, t \in (0,T)\}$, Γ be the boundary, which integrates the straight lines t = 0, t = T.

We choose a square grid and we write the area $\overline{D} = D + \Gamma$ by means of D_h, which is a set of points (x_i, t_j), the coordinates of which are:

$$x_i = ih, \ i = 0, \pm 1, \pm 2 ... \ h > 0$$

$$t_j = j\tau, \ j = 0,1...N, \ \tau > 0, \ N\tau \le T < (N+1)\tau.$$

We substitute $L(u)=f$ by the finite difference scheme (DS) $L_h\left(u^h\right)=f^h$. The exact solution of (1) in the grid knots $\left(x_i,t_j\right)$ is $u\left(x_i,t_j\right)$, and u_i^j is the corresponding approximate solution on the grid.

We have:

(5) $$\left.\frac{\partial u}{\partial t}\right|_{\left(x_i,t_j\right)}-\left.\frac{\partial^2 u}{\partial x^2}\right|_{\left(x_i,t_j\right)}=\left.\varphi(x,t)\right|_{\left(x_i,t_j\right)}, \qquad \begin{matrix} i=0,\pm1,\pm2,... \\ j=1,2,...N \end{matrix}$$

(6) $$\left.u(x,0)\right|_{\left(x_i,t_j\right)}=\left.\psi(x)\right|_{\left(x_i,t_j\right)}, \qquad i=0,\pm1...$$

From the formulas for numerical differentiation we have:

(7) $$\left.\frac{\partial u}{\partial t}\right|_{\left(x_i,t_j\right)}=\frac{u\left(x_i,t_{j+1}\right)-u\left(x_i,t_j\right)}{\tau}-\left.\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\right|_{\left(x_i,t_j^1\right)}, \qquad t_j<t_j^1<t_{j+1}$$

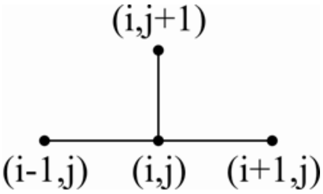
(8) $$\left.\frac{\partial u}{\partial t}\right|_{\left(x_i,t_j\right)}=\frac{u\left(x_i,t_j\right)-u\left(x_i,t_{j-1}\right)}{\tau}+\left.\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\right|_{\left(x_i,t_j^2\right)}, \qquad t_{j-1}<t_j^2<t_j$$

(9) $$\left.\frac{\partial u}{\partial t}\right|_{\left(x_i,t_j\right)}=\frac{u\left(x_i,t_{j+1}\right)-u\left(x_i,t_{j-1}\right)}{2\tau}-\left.\frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}\right|_{\left(x_i,t_j^3\right)}, \quad t_{j-1}<t_j^3<t_{j+1}$$

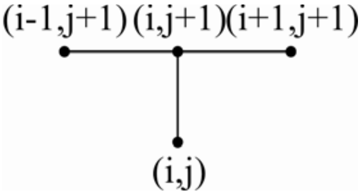
(10) $$\left.\frac{\partial^2 u}{\partial x^2}\right|_{\left(x_i,t_j\right)}=\frac{u\left(x_{i+1},t_j\right)-2u\left(x_i,t_j\right)+u\left(x_{i-1},t_j\right)}{h^2}-\left.\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\right|_{\left(x_i^1,t_j\right)}$$
$$x_{i-1}<x_i^1<x_{i+1}$$

We will call a **template** a set of knots, which are used when we replace $L(u)=f$ in the knot $\left(x_i,t_j\right)$ by DS $L_h\left(u^h\right)=f^h$.

The most widely used templates for parabolic equations are:

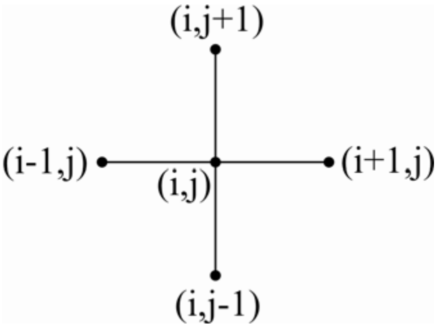


an obvious two-layered template



a non-obvious two-layered template

It is also possible to have a non-obvious three-layered template but it turns out that the scheme based on it is unstable for every h and τ.



Using (7) and (10) and the obvious template from (5) and (6), we obtain:

$$(11)\quad \begin{cases} \dfrac{u_i^{j+1}-u_i^j}{\tau}-\dfrac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{h^2}=\varphi_i^j+r_{ij}^{(1)}(h,\tau) \\ u_i^0=\psi_i, \qquad i=0,\pm1,\pm2,\ \ j=0,1,\ldots N-1 \end{cases}$$

$$r_{ij}^{(1)}(h,\tau)=-\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\bigg|_{\left(x_i,t_j^1\right)}-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_{\left(x_i^1,t_j\right)}$$ – an approximation error

Disregarding $r_{ij}^{(1)}$, we will get the obvious difference scheme (11).

By analogy, when a non-obvious two-layered template is used, as well as (9) and (10), we obtain the system:

$$(12)\quad \begin{vmatrix} \dfrac{u_i^{j+1}-u_i^j}{\tau}-\dfrac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2}=\varphi_i^j+r_{ij}^{(2)}(h,\tau) \\ u_i^0=\psi_i, \qquad i=0,\pm1,..., \ j=0,1,...N-1 \end{vmatrix}$$

$$r^{(2)}(h,\tau)=\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\bigg|_{(x_i,t_j^2)}-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_{(x_i^1,t_j)}$$ – an approximation error

Disregarding $r_{ij}^{(2)}$, we get the non-obvious scheme (12).

We will explain the order of approximation of (11) and (12) for a correlation between the time and space steps $\tau=\sigma h^s$, σ, s are positive numbers. If we presume that the following assessments are correct:

$$\max_{(x,t)\in D}\left|\frac{\partial^2 u}{\partial t^2}\right|\le M_2, \qquad \max_{(x,t)\in D}\left|\frac{\partial^4 u}{\partial x^4}\right|\le M_4,$$

then for the approximation error we will obtain:

$$(13)\quad \max_{ij}\left|r_{ij}^{(1)}(h)\right|\le\left(\frac{\sigma}{2}M_2+\frac{h^{2-s}}{12}M_4\right)h^s$$

$$(14)\quad \max_{ij}\left|r_{ij}^{(2)}(h)\right|\le\left(\frac{\sigma}{2}M_2+\frac{h^{2-s}}{12}M_4\right)h^s$$

There follows from the last formulas that the solution $u(x,y)$ of the problem $L(u)=f$ approximates with an error of the order s $(1\le s\le 2)$ with regard to h.

Realization of DS: The difference problem (11) can be solved in this way: for the meanings of the zero layer u_i^0, $i=0,\pm1,...$, a there are calculated u_i^1, $i=0,\pm1,...$, and in (11) $j=0$. Then for $j=1$ we will calculate u_i^2 and so on. When making such calculations the scheme is called **obvious**.

DS is not of this kind. If $j=0$, then in its left-hand side there will be a linear combination of $u_{i-1}^1, u_i^1, u_{i+1}^1, u_i^0$, and in its right-hand side - φ_i^0, ψ_i. In this case, in order to find the first layer $...u_{-2}^1, u_{-1}^1, u_0^1, u_1^1, u_2^1,...$, there is obtained an infinite system of linear equations and the scheme is called **non-obvious**. If the interval on x is not infinite but $x \in [a,b]$ and to the straight lines $x=a$ and $x=b$ are assigned conditions for the solution $u(x,t)$, then the obtained system is solved by the expulsion method.

DS (11) is stable for

(15) $\sigma = \frac{\tau}{h^2} \le \frac{1}{2}$.

This is a strong condition for τ, $\tau \le \frac{1}{2} h^2$.

DS (12) for conditions $u(a,t) = \gamma_0(t)$, $u(b,t) = \gamma_1(t)$ the stability condition is $\sigma = \frac{\tau}{h^2} > 0$, i. e. for arbitrary τ and h.

Conclusions:

1) For an obvious DS (11) the calculations of the next layer t are not hard to do but since τ is small enough, the layers on t are numerous.

2) For the non-obvious DS (12) more calculations of a layer on t are necessary but if there are no conditions for τ and h, the layers on t may not be numerous.

3) The scheme (11) approximates the problem (3)–(4) with an error $O(\tau + h^2)$ and is stable for $\sigma \le \frac{1}{2}$ therefore it is convergent. The approximate solution has an exactness of order $O(\tau + h^2)$.

Author: Lyuba Popova
PU „P. Hilendarski”