Introduction to finite difference method for solving partial differential equations

Partial differential equations (PDE) have a huge application in mathematics physics, hydrodynamics, acoustics and other scientific and application-oriented working areas. In most cases, however, they cannot be solved in an obvious way and that is why their approximate solution is widespread.

We will consider **the grid method** for solving **linear differential equations of the second order.** The building of different diagrams by means of the grid method depends on the type of the PDE and the kind of boundary conditions that they satisfy.

The most widely-spread partial solutions of linear PDE of the second order are:

1) Poisson's equation, which is an elliptic equation

$$\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

2) a parabolic equation, which is a heat-conductivity equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x, y)$$

3) a hyperbolic equation, which is a wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Some basic concepts of the theory of finite difference schemes (FDS) are the following:

I. Convergence and approximation of FDS

(1) $\quad L(u) = f$

(1) is a partial differential equation, where **u** is its solution in an area D, limited by a contour (boundary) Γ.

If $D_h = \{M_h\}$ are isolated points belonging to $\overline{D} = D + \Gamma$, then the number of the points in D_h depends on H (the smaller the h, the bigger their number is). D_h is called a **grid**, M_h are **grid knots**, and a function defined by the grid knots, is called a **grid function**.

If U is a space of functions continuous in D, and U_h is the space of grid functions defined in D_h, U is replaced by U_h by means of the grid method. If our task is to calculate $u_h(x,y)$, where $u(x,y)$ is the exact solution of (1), then $u_h(x,y)$ is a table of the values of $u(x,y)$ in the points of the grid D_h. As a rule it is not the exact solution that is found but $u^h \approx u_h(x,y)$, where u^h is called an approximate solution in points of the grid. In order to find it there is made a system of numerical equations.

(2) $L_h(u^h) = f^h$

L_h is a difference operator, corresponding to L, $f^h \in F^h$ (if $f(x,y) \in F$), we will call formula (2) **a difference scheme (DS).**

We will say that **(DS) is convergent** for $h \to 0$ if:

$\left\| u_h(x,y) - u^h \right\|_{U_h} \to 0$ for established grid standards $\|\bullet\|_{U_h}$, $\|\bullet\|_{F_h}$, corresponding to $\|\bullet\|_U$, $\|\bullet\|_F$ in the initial spaces U, F.

If

$\left\| u_h(x,y) - u^h \right\|_{U_h} \le Ch^s$, C – const, independent of $h, s > 0$, then the convergence is of order s with respect to h.

We will say that DS (2) **approximates** problem (1) if $L_h(u_h(x,y)) = f^h + \delta f^{(h)}$ and $\left\| \delta f^h \right\|_{F_h} \to 0$ for $h \to 0$.

δf^{h} is **an approximation error**. If $\left\| \delta f^{h} \right\|_{F_h} \leq Mh^{\sigma}$, M – const, independent of h, $\sigma > 0$, then DS (2) approximates problem (1) with an error of order σ regarding h.

II. Stability of DS:

The difference scheme (2) is called **stable** if $\exists h_0 > 0$ such that for all $h < h_0$ and for an arbitrary $f^{h} \in F_h$ there hold the following:

1) (2) has a single solution

2) $\left\| u^{h} \right\|_{U_h} \leq M \left\| f^{h} \right\|_{F_h}$, M – const, independent of h, f^{h} (the stability can be generalized as continuity with respect to the right-hand side).

The concept *stability* depends considerably on the established standards in U_h and F_h and it in some cases is possible to have condition (2) satisfied for some standards and not satisfied for others.

If for every sensible choice of standards condition (2) is not satisfied, then we say that DS is unstable.

So stability is an internal quality of DS and it can be influenced by altering some parameters, which take part in the scheme, or by changing DS as a whole.

We will quote a theorem, which combines the above mentioned concepts *approximation*, *convergence* and *stability*.

Theorem: Let the difference scheme $L_h\left(u^{h}\right) = f^{h}$ approximate the problem $L\left(u\right) = f$ for solution u(x,y) of order $s > 0$ regarding h and be stable. Then this DS is convergent and its order of convergence coincides with the order of approximation, i. e. $\left\| u_h\left(x, y\right) - u^{h} \right\|_{U_h} \leq Kh^{s}$, K – const, which does not depend on h (in broad terms from the approximation and stability $\Rightarrow$ convergence).

These concepts are considered in more detail in the following bibliography:

1. V. I. Krilov, V. V. Bobkov and P. I. Monastirnij, *Numerical methods*

2. A. A. Samarskij, *Introduction to the theory of finite difference schemes*

3. B. Sendov and V. Popov, *Numerical methods*.

Here are some common notes on studying the issues of composing, investigation and solving difference problems:

1) The rule for choosing a grid is specified, i. e. we substitute a continuous area for a discreet one. Most often the grid is square and uniform.

2) One or several DS are specified, approximation conditions are verified and the order of an approximation is established.

3) The stability of the built DS is proved, which is one of the important and complex questions. From 2) and 3) follows the convergence of DS by the quoted theorem.

4) There is considered the question of the numerical realization of DS. In the case of linear DS that is a system of linear algebraic equations. Even in the two-dimensional case the systems are of a great dimension and that poses problems in their realization. For that reason sometimes special methods are developed.

Further on we will consider the questions of the numeric solution of the different types of partial differential equations (PDE).

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