Examples on Variational Methods for Solving Boundary Problems for Ordinary Differential Equations (ODE) of the Second Order (Galerkin and Ritz methods)

Problem 1. Solve the following boundary problem with the Ritz method:

$$y'' + y + x = 0, \qquad y(0) = y(1) = 0,$$

by using the coordinate functions: $\varphi_0 = 0$, $\varphi_1 = x(1-x)$, $\varphi_2 = x^2(1-x)$. Compare the obtained approximate solution with the exact one $u(x) = \dfrac{\sin(x)}{\sin 1} - x$ in points from [0, 1].

Solution:

For the problem solved in (9), (see the references in the chapter "Variational methods – Galerkin and Ritz methods") we have: $p(x)=1,\ \ q(x)=-1,\ \ f(x)=-x$. The system (12), from which we will determine c_1, c_2, has the following form:

$$\left|\begin{array}{l} c_1\int_0^1 (\varphi_1'\varphi_1' - \varphi_1\varphi_1)dx + c_2\int_0^1 (\varphi_2'\varphi_1' - \varphi_2\varphi_1)dx = \int_0^1 x\varphi_1 dx \\ c_1\int_0^1 (\varphi_1'\varphi_2' - \varphi_1\varphi_2)dx + c_2\int_0^1 (\varphi_2'\varphi_2' - \varphi_2\varphi_2)dx = \int_0^1 x\varphi_2 dx \end{array}\right. .$$

Or by substitution: $\begin{pmatrix} \frac{3}{10} & \frac{3}{20} \\ \frac{3}{20} & \frac{13}{105} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} \\ \frac{1}{20} \end{pmatrix}$.

The solution to the system is c_1=0.19245 and c_2=0.17077. For the approximate solution we get

$$y = 0.19245x(1-x) + 0.17077x^2(1-x).$$

We compare the values of the approximate and the exact solution:

x	y	u
0.1	0.0189	0.0187
0.5	0.0695	0.0697
0.8	0.0527	0.0525

Problem 2. Solve the following boundary problem with Galerkin's method:

$$y'' + \frac{1}{x} y' = 0, \quad y(1) = 0, \quad y'(2) = \frac{1}{2},$$

by using this form for the solution: като решението се търси във вида $y = \frac{1}{2}(x-1) + c_1((x-2)^2 - 1) + c_2((x-2)^3 - 1)$ and compare it to the exact solution $u = \ln x$.

Solution:

Obviously $\varphi_0 = \frac{1}{2}(x-1)$, $\varphi_1 = ((x-2)^2 - 1)$, $\varphi_2 = ((x-2)^3 + 1)$ are the basis functions of the approximate solution, for which it can be verified that they satisfy the boundary conditions as it is described in **1.**

The system (6) has the following form:

$$\left| \begin{array}{l} c_1 \int_1^2 L\varphi_1 \varphi_1 dx + c_2 \int_1^2 L\varphi_2 \varphi_1 dx = \int_1^2 \left(f - L\varphi_0 \right).\varphi_1 dx \\ c_1 \int_1^2 L\varphi_1 \varphi_2 dx + c_2 \int_1^2 L\varphi_2 \varphi_2 dx = \int_1^2 \left(f - L\varphi_0 \right).\varphi_2 dx \end{array} \right. ,$$

where we calculate $L\varphi_0 = \frac{1}{2x}$, $L\varphi_1 = 2 + \frac{1}{x}.2(x-2)$, $L\varphi_2 = 6(x-2) + \frac{1}{x}.3(x-2)^2$.

We integrate the obtained coefficients of (7):

$$\left| \begin{array}{l} -0.9844 c_1 + 1.2033 c_2 = 0.2103 \\ 1.0748 c_1 - 1.4244 c_2 = -0.2407 \end{array} \right.$$

and we calculate the solution of the system: c_1=-0.0911, c_2=0.1002.

The approximate solution is:

$$y = \frac{1}{2}(x-1) - 0.0911((x-2)^2 - 1) + 0.1002((x-2)^3 - 1)$$

We compare the values of the approximate and the exact solution:

x	y	u
1.25	0.223	0.223
1.50	0.406	0.405
1.75	0.559	0.560

Problem 3. Solve the following boundary problem with the Ritz method

$$y'' + \frac{1}{x} y' = 0, \qquad y(1) = 0, \quad y(2) = \ln 2,$$

by using the respective coordinate functions of type (10) for $n = 2$.

Solution:

We are looking for the approximate solution $\mathrm{y}_2 = \varphi_0 + c_1\varphi_1 + c_2\varphi_2$, where from (10) we calculate $\varphi_0 = 0 + \frac{\ln 2 - 0}{2-1}(x-1) = \ln 2(x-1)$, , $\varphi_2 = (x-1)^2(2-x)$.

We write the right-hand side of the equation in a self-conjugated form after performing the necessary transformations: $y'' + \frac{1}{x} y' = 0 \ \rightarrow \ xy'' + y' = 0 \ \rightarrow \ (xy')' = 0$.

The system (12), from which we will determine c_1, c_2, for $p(x) = x$, $q(x) = 0$, $f(x) = 0$ is of the following type:

$$\left| \begin{array}{l} c_1 \int_1^2 x\varphi_1'\varphi_1' dx + c_2 \int_1^2 x\varphi_2'\varphi_1' dx = -\int_1^2 x\varphi_0\varphi_1 dx \\ c_1 \int_1^2 x\varphi_1'\varphi_2' dx + c_2 \int_1^2 x\varphi_2'\varphi_2' dx = -\int_1^2 x\varphi_0\varphi_2 dx \end{array} \right. .$$

In order to write it down in the following form $\sum_{i=1}^{2} A_{ik} c_i = -B_k$, $k = 1,2$ we calculate the coefficients:

$$A_{11} = \int_1^2 x\varphi_1'\varphi_1' dx = \int_1^2 x(3-2x)^2 dx = \frac{1}{2},$$

$$A_{21}=\int_1^2 x\varphi_1'\varphi_2'dx=\int_1^2 x(3-2x)(x-1)(5-3x)dx=\frac{17}{60},$$

$$A_{12}=A_{21},$$

$$A_{22}=\int_1^2 x\varphi_2'\varphi_2'dx=\int_1^2 x(x-1)^2(5-3x)^2dx=\frac{14}{60}=\frac{7}{30},$$

$$B_1=\int_1^2 x\varphi_0\varphi_1 dx=\int_1^2 x\ln 2(3-2x)dx=-\frac{\ln 2}{6},$$

$$B_2=\int_1^2 x\varphi_0\varphi_2 dx=\int_1^2 x\ln 2(x-1)(5-3x)dx=-\frac{\ln 2}{12}.$$

The solution of the system:

$$\left|\begin{array}{l}\frac{1}{2}c_1+\frac{17}{60}c_2=\frac{\ln 2}{6}\\ \frac{17}{60}c_1+\frac{7}{30}c_2=\frac{\ln 2}{12}\end{array}\right.$$

is: $c_1=0.2910,\ c_2=-0.1058$, and approximate solution is

$$y_2=\ln(2(x-1))+0.2910(x-1)(2-x)-0.1058(x-1)^2(2-x).$$

We compare the values of the approximate and the exact solution:

x	y	u
1.25	0.2228	0.2231
1.50	0.4055	0.4061
1.75	0.5596	0.5595

Problem 4. Solve the following boundary problem with the Ritz method

$$y''=x,\quad y(0)=0,\quad y(1)=1,$$

by using $\varphi_0,\varphi_1,\varphi_2$ from (10).

Solution:

We are looking the approximate solution in the following form $y=\varphi_0+c_1\varphi_1+c_2\varphi_2,\ c_1=?,\ c_2=?,$ where $\varphi_0=x,\quad \varphi_1=x(1-x),\quad \varphi_2=x^2(1-x),$ $p=1,\ f=x$.

After calculating the integrals, the system (12), from which we will determine

c_1, c_2, is:

$$\left|\begin{array}{l} \frac{1}{3}c_1 + \frac{1}{6}c_2 = -\frac{1}{12} \\ \frac{1}{6}c_1 + \frac{2}{15}c_2 = -\frac{1}{20} \end{array}\right.$$

and its solution is: $c_1 = -\frac{1}{6}, c_2 = -\frac{1}{6}$, or $y = x - \frac{1}{6}(x - x^3)$.

Problem 5. Solve the following boundary problem with Galerkin's method

$$y'' - 2xy' + 2y = x, \qquad y(0) = 0, \quad y'(1) = 1,$$

by using $\varphi_0, \varphi_1, \varphi_2$ following the formulas (10).

Solution:

We represent the approximate solution in the following form:

$y = \varphi_0 + c_1\varphi_1 + c_2\varphi_2$, $c_1 = ?$, $c_2 = ?$, where $\varphi_0 = x$, $\varphi_1 = x(x-2), \varphi_2 = x(x^2 - 3)$ are chosen in such a way that φ_0 satisfies the boundary conditions, and φ_1, φ_2 are zero boundary conditions, as it is described in **1.**

We calculate $L\varphi_0 = 0$, $L\varphi_1 = -2x^2 + 2$, $L\varphi_2 = 6x - 4x^3$ and the system (7) is:

$$\left|\begin{array}{l} c_1\int_0^1 L\varphi_1\varphi_1 dx + c_2\int_0^1 L\varphi_2\varphi_1 dx = \int_0^1 L\varphi_0\varphi_1 - f\varphi_1 dx \\ c_1\int_0^1 L\varphi_1\varphi_2 dx + c_2\int_0^1 L\varphi_2\varphi_2 dx = \int_0^1 L\varphi_0\varphi_2 - f\varphi_2 dx \end{array}\right.$$

After calculating the integrals we get:

$$\left|\begin{array}{l} -\frac{11}{15}c_1 - \frac{47}{30}c_2 = -\frac{5}{12} \\ -\frac{4}{3}c_1 - \frac{104}{35}c_2 = -\frac{4}{5} \end{array}\right. .$$

The solution of this system is: $c_1 = -0.169, c_2 = 0.34507$.

Problem 6. Find the approximate solution of the given boundary problem with Galerkin's method

$$y'' - y'\cos x + y\sin x = \sin x, \qquad y(-\pi) = y(\pi) = 2,$$

by choosing the following functions as a basis:

$\varphi_0 = 2,\ \varphi_1 = \sin x,\ \varphi_2 = \cos x + 1,\ \varphi_3 = \sin 2x,\ \varphi_4 = \cos 2x - 1.$

Solution:

We verify the conditions for the basis functions:

1. $\varphi_i,\ \ i = 0,...,4$ are linearly independent,

2. φ_0 satisfies the boundary conditions,

3. $\varphi_i,\ \ i = 0,...,4$ satisfy zero boundary conditions.

We look for the approximate solution in the following form $y = \varphi_0 + \sum_{i=1}^{4} c_i\varphi_i,\ c_i = ?$

We calculate

$$L\varphi_0 = 2\sin x,\quad L\varphi_1 = -\sin x - \cos 2x,\quad L\varphi_2 = \sin x - \cos x + \sin 2x,$$

$$L\varphi_3 = -4\sin 2x - \frac{1}{2}\cos x - \frac{3}{2}\cos 3x,\quad L\varphi_4 = -4\cos 2x + \frac{3}{2}\sin 3x - \frac{1}{2}\sin x.$$

We substitute in the system (7) and calculate its coefficients:

$$\left|\begin{array}{l} c_1 - c_2 \qquad + 0.5c_4 = 1 \\ \quad c_2 + 0.5c_3 \qquad = 0 \\ \quad c_2 - 4c_3 \qquad = 0 \\ c_1 + \qquad\quad + 4c_4 = 0 \end{array}\right. .$$

The solution of this system is: $c_1 = \frac{8}{7},\ c_2 = c_3 = 0,\ c_4 = -\frac{2}{7}$.

We compare the values of the approximate and the exact solution:

x	y	u
$-\pi/2$	1.429	1.368
0	2	2
$\pi/2$	3.714	3.718

Problem 7. Find the approximate solution of the given boundary problem with Galerkin's method

$$y'' - (x+1)y' - y = \frac{2}{(x+1)^3}, \quad y(0) = 1,\ y(1) = 0.5,$$

by choosing the following functions as a basis: $\varphi_0 = 1 - 0.5x$, $\varphi_1 = x(1-x)$, $\varphi_2 = x^2(1-x)$ and compare the approximate solution with the exact one $u = \dfrac{1}{x+1}$.

Solution:

We calculate $L\varphi_0 = x - 0.5$, $L\varphi_1 = 3x^2 - 3$, $L\varphi_2 = 4x^3 - 8x + 3$ and we substitute in the system (7), where

$$A_{11} = \int_0^1 L\varphi_1\varphi_1 dx = -0.35, \quad A_{21} = \int_0^1 L\varphi_2\varphi_1 dx = -0.2, \quad A_{12} = \int_0^1 L\varphi_1\varphi_2 dx = -0.15,$$

$$A_{22} = \int_0^1 L\varphi_2\varphi_2 dx = -0.138095, \quad B_1 = \int_0^1 f\varphi_1 - L\varphi_0\varphi_1 dx = 0.113706,$$

$$B_2 = \int_0^1 f\varphi_2 - L\varphi_0\varphi_2 dx = 0.0368441.$$

The solution of the system:

$$\left| \begin{array}{l} -0.35\, c_1 - 0.2\, c_2 = -0.113706 \\ -0.15 c_1 - 0.138095 c_2 = 0.0368441 \end{array} \right.$$

is $c_1 = -0.454552, c_2 = 0.226935$.

We compare: $y = 1 - 0.5x + c_1 x(1-x) + c_2 x^2 (1-x)$ and $u = \dfrac{1}{x+1}$ in the points $x =$ 0.2, 0.5, 0.7:

$$\begin{array}{ll} y(0.2) = 0.83453, & u(0.2) = 0.83333 \\ y(0.5) = 0.66473, & u(0.5) = 0.66667. \\ y(0.7) = 0.58790, & u(0.7) = 0.58824 \end{array}$$

Problem 8. Solve the following boundary problem with the Ritz method:

$((x+1)y')' = ax + b, y(0) = y(1) = 0$, where *a*, *b* are parameters, taking values from 0 to 9.

Solution:

We are looking for $y = \varphi_0 + c_1\varphi_1 + c_2\varphi_2$, $c_1 = ?$, $c_2 = ?$ for $\varphi_0 = 0$, $\varphi_1 = x(1-x)$, $\varphi_2 = x^2(1-x)$. In this case $p = (1+x)$, $q = 0$, $f = ax + b$. In the system (12), from which we will determine c_1, c_2, after calculating the integrals we obtain:

$$A_{11} = \int_0^1 p\varphi_1'\varphi_1'dx = \frac{1}{2}, \quad A_{12} = A_{21} = \int_0^1 p\varphi_1'\varphi_2'dx = \frac{17}{60},$$

$$B_1 = \int_0^1 f\varphi_1 dx = \frac{a+2b}{12}, \quad B_2 = \int_0^1 f\varphi_2 dx = \frac{3a+5b}{60}.$$

The solution of the system:

$$\left| \begin{array}{l} A_{11}c_1 + A_{21}\ c_2 = -B_1 \\ A_{12}c_1 + A_{22}c_2 = -B_2 \end{array} \right.$$

are coefficients in the approximate solution.

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