# Modified Euler method

As in the previous Euler method, we assume that the following problem (Cauchy problem) is being solved:

$$(1) \qquad \begin{aligned} & y'(x) = f(x,y) \quad , \quad x \in [a,b] \\ & y(a) = y_0 \end{aligned} .$$

Once more we will use an uniform mesh along the axel x with a step of $h = \frac{b-a}{n}$:

$$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b, \text{ where } x_i = a + ih, \quad i = 0, 1, \dots, n \quad .$$

Approximated values of the function y(x) in the points $x_i, \quad i = 0, 1, 2, \dots, n$ are found using the initial given value $y_0$ , in other words the solution is in the form of a table of values

$y_1, y_2, \dots, y_n$ .

The modified Euler method is also an explicit method, but calculating each next value of the solution $y_{i+1}$ is done with the help of two consecutive calculations of the formulas:

$$(2) \qquad \begin{aligned} & f_{i+\frac{1}{2}} = f(x_i + \frac{h}{2}, y_i + \frac{h}{2} f_i) \\ & y_{i+1} = y_i + h f_{i+\frac{1}{2}} \; , \qquad i = 0, 1, \dots, n-1 \end{aligned} .$$

The local error of the modified Euler method is one order higher than the local error of the usual Euler method, in other words $r_i = O(h^3)$, with an assumption for the third derivative of the solution.

**Example 1**. Solve the problem numerically using the modified Euler method and compare the result with the exact solution $y^*(x)$ :

$$y' = \frac{2y}{x} + x \; , \quad x \in [1, 1.4], \quad y(1) = 1 \, , \quad n = 4, \quad y^*(x) = x^2 + x^2 \ln(x) \; .$$

| $i$ | $x_i$ | $y_i$ | $y^*_i$ |
|---|---|---|---|
| 0 | 1.0 | 1.00000 | 1.00000 |
| 1 | 1.1 | 1.32405 | 1.32533 |
| 2 | 1.2 | 1.69982 | 1.70254 |
| 3 | 1.3 | 2.12905 | 2.13340 |
| 4 | 1.4 | 2.61336 | 2.61949 |

Table 4 – solutions to Example 1.

**Task 1**. Apply the modified Euler method to derive an approximated solution to the following problems:

a) $y' = \text{Exp}(-\frac{x}{y})\,(\frac{x}{y} - 1)$ , $x \in [0, 1]$, $y(0) = 1$ , $n = 5$

b) $y' = x\,\text{Cos}(y) - 2\,y\,\text{Sin}(x)$ , $x \in [0, \frac{\pi}{2}]$, $y(0) = -1$ , $n = 3$

c) $y' = y + \frac{x}{y}$ , $x \in [-1, 0]$, $y(-1) = 1$ , $n = 5$

d) $y' = \frac{2y - x}{x + y}$ , $x \in [2, 3]$, $y(2) = 0.5$ , $n = 5$

e) $y' = \text{Sin}\left(x^2 + y^2\right)$, $x \in [0, 0.3]$, $y(0) = 0$ , $n = 3$

**Task 2**. Integrate the problem using the modified Euler method:

$$y' = \alpha x^2 + \beta y^2 \ , \quad x \in [0, 0.8], \quad y(0) = 0 , \quad n = 4 \ ,$$

Where the coefficient values are $\alpha = 1, 2, 3$ and $\beta = 1, 2, 3$ .

**Task 3**. Solve the problem with the help of the modified Euler method:

$$y' = \frac{\alpha}{x^2 + y^2 + 1} \ , \quad x \in [0, 0.4], \quad y(0) = 0 , \quad n = 4, \quad \alpha = 1,\ 1.1,\ 1.2$$

**Task 4**. Solve the problems using the modified Euler method:

a) $\begin{cases} y' = y - z \\ z' = x^2 + \frac{y}{z} \end{cases}$ , $0 \le x \le 0.3$, $y(0) = 1$ , $z(0) = 2$ , $n = 3$

b) $\begin{cases} y' = \alpha + y + z \\ z' = \frac{2z}{y + \beta} \end{cases}$ , $1 \le x \le 1.2$, $y(0) = 1$ , $z(0) = -1$ , $n = 2$

$\alpha = 0,\ 0.1,\ 0.2,\ 0.3$ , $\beta = 1,\ 2,\ 3.$

Author: Snezhana G. Gocheva-Ilieva

Plovdiv University