Euler-Cauchy method

The following problem is solved (Cauchy problem):

(1) $$\begin{array}{l} y'(x) = f(x,y) \quad , \quad x \in [a,b] \\ y(a) = y_0 \end{array}.$$

It is used again an uniform mesh in the given interval with a step of $h = \dfrac{b-a}{n}$:

$$x_0 = a < x_1 < x_2 < \ldots < x_{n-1} < x_n = b \text{,where } x_i = a + ih, \quad i = 0, 1, \ldots, n \;.$$

Using the initial value y_0 to find the approximated values of the function $y(x)$ in the given points x_i, $i = 0, 1, 2, \ldots, n$, in other words, the solution will be obtained in the form of a table of values $y_1, y_2, \ldots, y_n$.

With this method an approximated y_{i+1} is found using a given y_i in two steps – foreseeing of the initial approximation (by a predictor formulae) $y_{i+1}^{(0)}$ and adjusting iteratively of the value (by a corrector formulae), which is accepted as y_{i+1}, according to the formulas:

(2) $$\begin{array}{l} y_{i+1}^{(0)} = y_i + h f_i \\ y_{i+1} = y_i + \dfrac{h}{2}(f_i + f(x_{i+1}, y_{i+1}^{(0)})), \quad i = 0, 1, \ldots, n-1 \end{array}$$

The local theoretical error of the Euler-Cauchy method is $r_i = O(h^3)$. In practice different ways for its evaluation are used with each specific problem. For example we calculate two approximations: y_i with a step of h and $\tilde{y}_i$ with a halved step: $\tilde{h} = \dfrac{h}{2}$, after which the error is evaluated using the formula:

(3) $$|\, y(x_i) - \tilde{y}_i \,| \approx \frac{1}{3}|y_i - \tilde{y}_i|$$

If the results do not satisfy us we halve the step, repeat the calculations and so on.

Task 1. Find the approximated solutions to the following problems for the indicated intervals, initial values and number of divisions n using the Euler-Cauchy method:

a) $y' = \sqrt{1 + \dfrac{y}{x}}$, $\quad x \in [1, 1.3]$, $\quad y(1) = 1$, $\quad n = 3$

b) $y' = x + y + y^2$, $\quad x \in [-1, -0.5]$, $\quad y(-1) = 0$, $\quad n = 5$

c) $y' = \dfrac{\ln(x)}{x + y}$, $\quad x \in [1, 2]$, $\quad y(1) = 1$, $\quad n = 5$

d) $y' = \mathrm{Cos}(x + y) - \mathrm{Sin}(x - y)$, $x \in [1, 2]$, $y(1) = 1$, $n = 5$

e) $y' = \dfrac{x}{\mathrm{Cos}(y)} + \dfrac{y}{\mathrm{Sin}(x)}$, $x \in [1, 1.4]$, $y(1) = 0$, $n = 4$

f) $y' = \dfrac{x + \alpha}{2 - \mathrm{Sin}(x + y)}$, $x \in [1, 2]$, $y(1) = 1$, $n = 4$

$\alpha = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3.$

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