Euler method for numerical solving of ordinary differential equations and systems

We will solve the initial problem:

$$(1)\qquad \begin{aligned} & y'(x) = f(x,y) \quad , \quad x \in [a,b] \\ & y(a) = y_0 \end{aligned} .$$

Let the interval $[a,b]$ be divided by *n* equal in length subintervals with the help of a constant step $h = \frac{b-a}{n}$ in a way that gives us the points:

$$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b, \quad x_i = a + ih, \quad i = 0, 1, \dots, n \ .$$

Approximated values of the function $y(x)$ in the upper points x_i, $i = 0, 1, 2, \dots, n$ at a given initial value y_0 are sought. In other words the solution is seeking in the form of a table of values $y_1, y_2, \dots, y_n$.

The Euler method is one of the simplest one step methods that use a given y_0 to find an approximated value of the solution y_1 in the next point x_1; using y_1 we find y_2 by following the same procedure and so on until y_n.

In order to solve problem (1) the approximated values of the unknown function are calculated using the recurrent formula:

$$(2)\qquad y_{i+1} = y_i + h f(x_i, y_i), \qquad i = 0, 1, 2, \dots, n-1 \ .$$

Assuming that the function $f(x,y)$ is continuous and has bounded partial derivatives of the first order with respect to x and y in $[a,b]$ such that

$$(3)\qquad |f| \le M_1, \quad \left|\frac{\partial f}{\partial x}\right| \le M_2 \ , \quad \left|\frac{\partial f}{\partial y}\right| \le M_3 \ ,$$

it can be shown that the Euler method has a relatively high local error:

$$(4)\qquad r_i = |\, y(x_i) - y_i \,| \le C h^2 \quad \text{or} \quad r_i = O(h^2) \ ,$$

where $C = \frac{M_2 + M_1 M_3}{2}$ is a constant that is independent from the step *h*. In short, the local error of the Euler method is proportional to h^2 at a slowly increasing right side of the equation (1) and its corresponding partial derivatives.

The following evaluation is true for the global theoretical error in the interval:

$$(4)\qquad r(x_i) = |\, y(x_i) - y_i \,| \le \overline{C} h \quad \text{или} \quad r = O(h) \ .$$

In the case of systems of equations formula (2) is written for every vector coordinate. For example for the initial system of two equations

$$(5)\quad \begin{cases} y' = f(x,y,z) \\ z' = g(x,y,z) \end{cases}, \quad a \le x \le b$$

$$y(a) = y_0, \qquad z(a) = z_0$$

the corresponding formulas for the serial calculation of y_{i+1}, z_{i+1} are:

$$(6)\quad \begin{cases} y_{i+1} = y_i + hf(x_i, y_i, z_i) \\ z_{i+1} = z_i + hg(x_i, y_i, z_i) \end{cases}, \qquad i = 0, 1, 2, \dots, n-1$$

Task 1. Using the Euler method find an approximated solution $y(x)$ to the following initial problems at different values of the step h. Compare the results with the exact solutions $y^*(x)$.

a) $y' = 0.2y$, $y(0) = 1$, $x \in [0, 0.5]$, $h = 0.1$; $y^*(x) = \exp(0.2x)$

b) $y' = 10y$, $y(0) = 1$, $x \in [0, 0.5]$, $h = 0.1$ и $h = 0.05$; $y^*(x) = \exp(10x)$.

Solution: a) Since h=0.1, a=0 and b=0.5, then n=5 and the integration points are $x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4,\ x_5 = 0.5$, respectively. We know that $y_0 = 1$. Using formula (2) we calculate

$$y_1 = y_0 + 0.2y_0 = 1 + 0.2 = 1.02,$$

after that analogically y_2 and so on. The derived results, as well as the exact solutions to the problem, are shown in table 1. Irrelevant of the accuracy of the intermediate calculations according to (3) the method secures a small accuracy. For the examined problem at h=0.1 the theoretical evaluation of the error is $r_i \approx 0.01$. Therefore with a slow change to the solution as an end result we take the values of y with two symbols after the decimal point, i.e. y(0.5)=1.10. Compare with the exact solution.

i	x_i	$f(x_i, y_i)$	y_i	y^*_i
0	0.0	0.2	1	1.000000
1	0.1	0.204	1.02	1.020201
2	0.2	0.20808	1.0404	1.040811
3	0.3	0.212242	1.06121	1.061837
4	0.4	0.216486	1.08243	1.083287
5	0.5	-	1.10408	1.105171

Table 1 – Solution of the problem $y' = 0.2*y$, $y(0) = 1$ using the Euler method with a step h=0.1.

b) Although the equation is similar to a) we can see that the solution is different from the exact one as the argument rises (table 2). The error of the method increases too quickly due to the derivatives (4) being large. This type of problem relates to the so called **stiff differential problems**. Satisfactory results can be derived, for example, using a small enough step. The values of the solution are shown on table 3 with the help of a computer using a step of h=0.001 and h=0.000001.

i	x_i	$f(x_i, y_i)$ при h=0.1	y_i при h=0.1	$f(x_i, y_i)$ при h=0.05	y_i при h=0.05	y^*_i
0	0.0	10	1.000	10.0000	1.00000	1.000000
1	0.1	20	2.000	22.5000	2.25000	2.718282
2	0.2	40	4.000	50.6250	5.06250	7.389056
3	0.3	80	8.000	113.9060	11.39060	20.085540
4	0.4	1600	16.000	256.2890	25.62890	54.598150
5	0.5	-	32.000	-	57.66500	148.413200

Table 2 – Solutions to the problem y' = 10y, y(0) = 1, calculated using the Euler method with steps h=0.1 and h=0.05 .

i	x_i	y_i h=0.001	y_i h=0.000001	y^*_i
0	0.0	1.00000	1.00000	1.000000
1	0.1	2.70481	2.71823	2.718282
2	0.2	7.31602	7.38880	7.389056
3	0.3	19.78851	20.08440	20.085540
4	0.4	53.52412	54.59430	54.598150
5	0.5	144.77304	148.40000	148.413200

Table.3 – Solutions of the problem y' = 10y, y(0) = 1 using the Euler method with a step of h=0.001 and h=0.000001, printed in the necessary points

Task 2. Using the Euler method find the approximated solution $y(x)$ to the following problem with the given values of the step h. Compare the results with the exact solution $y^*(x)$.

$$y' = y - 2\sin(x), \quad y(0) = 1, \quad x \in [0, 0.5], \quad h = 0.1 \text{ и } h = 0.05, \quad y^*(x) = \cos(x) + \sin(x)$$

Task 3. Solve the following problems using the Euler method and compare the derived results to the exact solutions $y^*(x)$:

a) $y' = \dfrac{y}{x} + 1, \quad y(1) = 0, \quad x \in [1,2], \quad n = 10, \quad y^*(x) = x\ln(x)$

b) $y' = \dfrac{y-\alpha}{x} + 1, \quad y(1) = \alpha, \quad \alpha = 1, 2, 3; \quad x \in [1,2], \quad n = 5, \quad y^*(x) = x\ln(x) + \alpha$.

Task 4. Solve the following problems using the Euler method:

a) $y' = \dfrac{x+y}{x^2 - y^2}, \quad x \in [1, 2], \quad y(1) = 1, \quad n = 5$

b) $y' = \alpha \dfrac{x - y}{x + y}, \quad x \in [0, 1], \quad y(0) = \alpha > 0, \quad n = 10$

c) $y' = x + \sqrt{x^2 + y^2}, \quad x \in [0, 0.5], \quad y(0) = 0.1, \quad n = 5$

d) $y' = x \ln(y), \quad x \in [1, 2], \quad y(0) = 2, \quad n = 10$

e) $y' = \cos(x) - \alpha \sin(y + x), \ \alpha = 1.1,\ 1.2,\ 1.3, \quad x \in [2, 3], \quad y(2) = 1, \quad n = 5$

Task 5. Check the stability of the Euler method in the solutions of the following problems. Find out at what value of the step h there is a guaranteed error of approximation $\varepsilon = 0.001$.

a) $y' = 3y - 1, \quad x \in [1, 2], \quad y(1) = 0$

b) $y' = -0.5\, y + 2, \quad x \in [0, 2], \quad y(0) = 0$

c) $y' = -100\, y, \quad x \in [0, 10], \quad y(0) = 0$

Task 6. Apply the Euler method in order to integrate the problems:

a) $y'' = 3y' + 5xy + \alpha, \quad \alpha = 0, 1, 2, 3, \quad x \in [0, 0.3], \quad y(0) = 1, \quad y'(0) = -1, \quad h = 0.1$

b) $y'' = x\, y' + y^2, \quad x \in [1, 1.3], \quad y(1) = 0, \quad y'(1) = 1, \quad h = 0.1$

c) $\begin{cases} y' = x - yz \\ z' = 5x - z \end{cases}, \quad 0 \le x \le 0.4, \quad y(0) = 1, \quad z(0) = 0, \quad n = 4$

d) $\begin{cases} y' = \dfrac{y}{2} + z \\ z' = y - \dfrac{z}{2} \end{cases}, \quad 0 \le x \le 1, \quad y(0) = 1, \quad z(0) = 0, \quad n = 5$.

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