INTRODUCTION TO THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

The ordinary differential equations are a powerful mathematical instrument for description and modeling of phenomena and laws in different fields of science, engineering, economics, warfare etc. Deriving a solution in the form of a formula (i.e. in analytical form) is possible only for a few types of equations. At the same time a large number of methods for numerically solving of a given differential problem have been developed for a wide range of equation classes.

Classical methods for solutions of the so-called Cauchy problem or the initial problem for ordinary differential equations are presented in this section.

Definition 1. We call an *ordinary differential equation of order* k the following equation:

(1) $$F(x, y(x), y'(x), \dots, y^{(k)}(x)) = 0 \quad ,$$

where x is an independent variable, F - a given function and the function y(x) and its derivatives up to order k are the unknown quantities. When the equation is solved regarding the higher derivative, namely:

(2) $$y^{(k)}(x) = f(x, y(x), \dots, y^{(k-1)}(x))$$

we will say that it is in a *normal form*. Further, we will consider the equation of the first order in the form:

(3) $$y'(x) = f(x, y)$$

Definition 2. We call *solutions* (or *integrals*) of equation (1) ((2) and (3), respectively) any function y(x), which, along with its necessary number of derivatives, which substituted in the equation, satisfy it for all the values of the independent variable x in a finite or infinite interval [a, b].

The general solution of an ordinary differential equation of order k looks like this:

(4) $$y = y(x, C_1, C_2, \dots, C_k),$$

where $C_1, C_2, \dots, C_k$ are arbitrary constants. Setting a given number of constants will result in some *particular solution*.

Definition 3. *The Cauchy problem* (also the *initial problem*) for equation (1) consists of defining a particular solution which satisfies the equation as well as *k* initial conditions, set in point $x_0 \in [a, b]$:

(5) $$y(x_0) = y_0, \quad y'(x_0) = y'_0, \; \dots, \; y^{(k-1)}(x_0) = y_0^{(k-1)} \; .$$

The Cauchy problem for systems of p ordinary differential equations:

$$
(6) \qquad \begin{cases} y_1'(x) = f_1(x, y_1, y_2, \dots, y_p) \\ y_2'(x) = f_2(x, y_1, y_2, \dots, y_p) \\ \quad \dots\dots\dots \\ y_p'(x) = f_p(x, y_1, y_2, \dots, y_p) \end{cases}
$$

is broken down to finding the functions $y_1(x),\ y_2(x), \dots, y_p(x)$ which satisfy this system and the initial conditions:

(7) $\qquad y_1(x_0) = y_{10},\ \ y_2(x_0) = y_{20},\ \dots,\ \ y_p(x_0) = y_{p0}$.

This system can be written in vector form analogical to (3), namely:

$$
(8) \qquad \begin{aligned} \vec{y}'(x) &= \vec{f}(x, \vec{y}) \\ \vec{y}(x_0) &= \vec{y}_0 \end{aligned} \qquad ,
$$

where $\vec{y}'(x) = (y_1, y_2, \dots, y_p)$, $\qquad \vec{f} = (f_1, f_2, \dots, f_p)$.

The ordinary differential equation (2) can be converted to a system like (6) or (7) with the help of a suitable sustitution.

From here on, assuming nothing else has been said, we will consider the initial problem:

$$
(9) \qquad \begin{aligned} y'(x) &= f(x, y) \quad , \qquad x \in [a, b] \\ y(a) &= y_0 \end{aligned} \qquad .
$$

We assume that the conditions for the existence of an only solution have been met. Besides that some methods require that the solution is stable, in other words little errors in the initial data will lead only to little result errors.

During numerical solution finding of (9) the interval [a, b] is broken down to *n* equal subintervals with the help of a constant step $h = \dfrac{b-a}{n}$. Using that we get the points:

$$
x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b, \text{ where } x_i = a + ih, \quad i = 0, 1, \dots, n
$$

are called knots or points of division. Approximated values of the function y(x) in the chosen points $x_i,\ \ i = 0, 1, 2, \dots, n$ are found using the initial value y_0, in other words the solution is in the form of a table of values $y_1, y_2, \dots, y_n$.

When there is a need for analytical presentation, the table data can be approximated afterwards with the help of relevantly picked methods. On the other hand, the advantages of the numerical methods are the relative simplicity and monotonic solution of a wide range of problems and solving with accuracy set in advance in an arbitrary number of points of the interval.

Therefore numerical methods do not help in finding the general solution, but only the particular solution in the form of the table. They can be used to solve more complicated problems compared to analytical methods.

Author: Snezhana G. Gocheva-Ilieva
University of Plovdiv