Preparation of system of linear algebraic equations with positive defined matrix in order to apply the iterative approximation method

(1) $Ax = b$, x^* - root $\leftrightarrow$ $\lambda Ax = \lambda b$, $\lambda \neq 0$ $\leftrightarrow$ $x = x - \lambda Ax + \lambda b$ $\leftrightarrow$

(2) $\boxed{x = (E - \lambda A)x + \lambda b}$,

Let us assume the matrix A is **positive defined** and ν is any one of its norm. If we choose $\lambda = 2/\rho$, where $\rho > \nu$, then the equation (2) will assume the form:

(3) $x = \left(E - \dfrac{2}{\rho}A\right)x + \dfrac{2}{\rho}b$

In this case formula (3) induces the iteration process

(4) $x^{(k)} = \left(E - \dfrac{2}{\rho}A\right)x^{(k-1)} + \dfrac{2}{\rho}b$, $k = 1, 2, \dots$,

which converges to the solution x^* of system (1).

> Take note, that in this case it is not necessary to check the sufficient condition for the convergence of iterative method with the matrix $(E - \dfrac{2}{\rho}A)$.

Algorithm

$\boxed{1.}$ Obtaining the matrix $E - \dfrac{2}{\rho}A$ and the vector $\dfrac{2}{\rho}b$.

$\boxed{2.}$ Execution of the iteration process (4) until reaching the desired accuracy ε through the stop criteria:

If $\left|x_i^{(k)} - x_i^{(k-1)}\right| < \varepsilon$, for $\forall i = \overline{1,n}$ then $x_i^* = x_i^{(k)}$ with an accuracy of ε.

Example. It is known for the system below that its matrix is positive defined. Prepare the system for the solution using formula (4). Make three iterations at $x^{(0)} = (0,0,0)$.

$$\left|\begin{array}{llll} 9x_1 & +3x_2 & & =12 \\ 3x_1 & +10x_2 & +6x_3 & =-8 \\ & 6x_2 & +13x_3 & =1 \end{array}\right. .$$

Solution:

1. $A=\begin{pmatrix}9 & 3 & 0\\ 3 & 10 & 6\\ 0 & 6 & 13\end{pmatrix} \rightarrow \|A\|_1 = 19 \rightarrow$ we choose $\rho = 20\ (\rho > \nu)$.

Therefore:

$$\lambda = \frac{2}{\rho} = \frac{2}{20} = 0{,}1 \rightarrow E - \frac{2}{\rho}A = \begin{pmatrix}0{,}1 & -0{,}3 & 0\\ -0{,}3 & 0 & -0{,}6\\ 0 & -0{,}6 & -0{,}3\end{pmatrix};\ b = \begin{pmatrix}12\\ -8\\ 1\end{pmatrix} \rightarrow \frac{2}{\rho}b = \begin{pmatrix}1{,}2\\ -0{,}8\\ 0{,}1\end{pmatrix}.$$

2. Execution of three iterations:

$$\left|\begin{array}{llllll} x_1^{(1)} = & 0{,}1x_1^{(0)} & -0{,}3x_2^{(0)} & +1{,}2 & =0{,}1.0-0{,}3.0\ +1{,}2=1{,}2 \\ x_2^{(1)} = & -0{,}3x_1^{(0)} & -0{,}6x_3^{(0)} & -0{,}8 & =-0{,}3.0\ -0{,}6.0-0{,}8=-0{,}8 \\ x_3^{(1)} = & -0{,}6x_2^{(0)} & -0{,}3x_3^{(0)} & +0{,}1 & =-0{,}6.0-0{,}3.0+0{,}1=0{,}1 \end{array}\right. \rightarrow \boxed{\begin{array}{l} x_1^{(1)} = 1{,}2 \\ x_2^{(1)} = -0{,}8 \\ x_3^{(1)} = 0{,}1 \end{array}}$$

$$\left|\begin{array}{llllll} x_1^{(2)} = & 0{,}1x_1^{(1)} & -0{,}3x_2^{(1)} & +1{,}2 & =0{,}1.1{,}2-0{,}3.(-0{,}8)+1{,}2=1{,}56 \\ x_2^{(2)} = & -0{,}3x_1^{(1)} & -0{,}6x_3^{(1)} & -0{,}8 & =-0{,}3.1{,}2\ -0{,}6.0{,}1-0{,}8=-1{,}22 \\ x_3^{(2)} = & -0{,}6x_2^{(1)} & -0{,}3x_3^{(1)} & +0{,}1 & =-0{,}6.(-0{,}8)-0{,}3.0{,}1+0{,}1=0{,}55 \end{array}\right. \rightarrow \boxed{\begin{array}{l} x_1^{(2)} = 1{,}56 \\ x_2^{(2)} = -1{,}22 \\ x_3^{(2)} = 0{,}55 \end{array}}.$$

Do the third iteration individually. The results are in the table below.

k \ $x^{(k)}$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	1,2	-0,8	0,1
2	1,56	-1,22	0,55
3	1,722	-1,598	0,667
…..	...	...	...
x^*	2	-2	1

Comment. It is obvious that we have a variety of choices for $\rho\ (\rho > \nu)$ and from here for $\lambda = 2/\rho$. Then a question arises – is there a way to choose such a ρ, so that formula (4) works best. Sadly there is still no answer to that question.

Author: Ilia Markelov, ilmak@pu.acad.bg