Matrix inversion

Problem formulation

A quadratic matrix A from *n-th* row is given:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots\dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

The inverse matrix A^{-1} is sought, i.e. such a matrix that $AA^{-1} = E$. We denote

$$A^{-1} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ & \dots\dots & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Finding the inverse matrix by columns is reduced to solving n systems of linear algebra equations with one and the same matrix A and different right sides equal to the columns of a singular matrix E. These problems are:

$$A\begin{pmatrix} x_{11} \\ x_{21} \\ \dots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} x_{12} \\ x_{22} \\ \dots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \ \dots\ , A\begin{pmatrix} x_{1n} \\ x_{2n} \\ \dots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}.$$

It is convenient to use the Gauss-Jordan method which transforms the expanded matrix into a singular one, the right side containing the solutions.

Algorithm

$$\left(\begin{array}{c|cccc} & 1 & 0 & \dots & 0 \\ A & 0 & 1 & \dots & 0 \\ & & & \dots & \\ & 0 & 0 & \dots & 1 \end{array}\right) \leftrightarrow \dots \leftrightarrow \left(\begin{array}{c|cccc} & x_{11} & x_{12} & \dots & x_{1n} \\ E & x_{21} & x_{22} & \dots & x_{2n} \\ & & & \dots & \\ & x_{n1} & x_{n2} & \dots & x_{nn} \end{array}\right)$$

Example. Invert matrix A and find its determinant using Gauss-Jordan's method choosing a pivot element from a column:

$$A = \begin{pmatrix} 3 & 5 & 2 \\ 1 & 0 & 6 \\ 2 & 1 & 4 \end{pmatrix}.$$

Solution:

$$\left(\begin{array}{ccc|ccc} \boxed{3} & 5 & 2 & 1 & 0 & 0 \\ 1 & 0 & 6 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) : (3) \quad \leftrightarrow \quad \left(\begin{array}{ccc|ccc} 1 & \frac{5}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 0 & 6 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \; .(-1) \;\; .(-2) \quad \leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{5}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{5}{3} & \frac{16}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \updownarrow \quad \leftrightarrow \quad \left(\begin{array}{ccc|ccc} 1 & \frac{5}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \boxed{-7/3} & \frac{8}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & -\frac{5}{3} & \frac{16}{3} & -\frac{1}{3} & 1 & 0 \end{array}\right) : (-\tfrac{7}{3}) \quad \leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{5}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{8}{7} & \frac{2}{7} & 0 & -\frac{3}{7} \\ 0 & -\frac{5}{3} & \frac{16}{3} & -\frac{1}{3} & 1 & 0 \end{array}\right) \; .(\tfrac{5}{3}) \;\; .(-\tfrac{5}{3}) \quad \leftrightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{18}{7} & -\frac{1}{7} & 0 & \frac{5}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{2}{7} & 0 & -\frac{3}{7} \\ 0 & 0 & \boxed{24/7} & \frac{1}{7} & 1 & -\frac{5}{7} \end{array}\right) : (\tfrac{24}{7}) \quad \leftrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{18}{7} & -\frac{1}{7} & 0 & \frac{5}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{2}{7} & 0 & -\frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{24} & \frac{7}{24} & -\frac{5}{24} \end{array}\right) \; .(\tfrac{8}{7}) \;\; .(-\tfrac{18}{7}) \quad \leftrightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & -\frac{3}{4} & \frac{5}{4} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{24} & \frac{7}{24} & -\frac{5}{24} \end{array}\right) \rightarrow$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} & \frac{5}{4} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{24} & \frac{7}{24} & -\frac{5}{24} \end{pmatrix}.$$

It is obvious that the pivot elements are 3, $-\frac{7}{3}$ and $\frac{24}{7}$. We also have a change in two rows (a multiplier (-1) appears in front of the product of the pivot elements). Consequently $\det A = -3.\left(-\frac{7}{3}\right).\left(\frac{24}{7}\right) = 24$.

By I. Makrelov, ilmak@uni-plovdiv.bg