Example of coefficient instability of the Gauss elimination method for solving linear system of algebraic equations

If a choice of the pivoting element in the Gauss elimination method is not applied there can be a loss of accuracy. In the next example we assume that we are working with a hypothetical machine which realizes arithmetic by a floating-point system in base 2 and 5 significant digits. Let us solve the system below at these conditions using the Gauss method without a choice of a pivoting element. The exact solution is (0, -1, 1).

$$\left|\begin{array}{llll} 10x_1 & -7x_2 & & =7 \\ -3x_1 & +2{,}099x_2 & +6x_3 & =3{,}901 \\ 5x_1 & -x_2 & +5x_3 & =6 \end{array}\right. .$$

Solution:

$$\left(\begin{array}{ccc|c} 10 & -7 & 0 & 7 \\ -3 & 2{,}099 & 6 & 3{,}901 \\ 5 & -1 & 5 & 6 \end{array}\right) :(10) \leftrightarrow \left(\begin{array}{ccc|c} 1 & -0{,}7 & 0 & 0{,}7 \\ -3 & 2{,}099 & 6 & 3{,}901 \\ 5 & -1 & 5 & 6 \end{array}\right) \begin{array}{cc} (3) & .(-5) \end{array} \leftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -0{,}7 & 0 & 0{,}7 \\ 0 & -10^{-3} & 6 & 6{,}001 \\ 0 & 2{,}5 & 5 & 2{,}5 \end{array}\right) :(-10^{-3}) \leftrightarrow \left(\begin{array}{ccc|c} 1 & -0{,}7 & 0 & 0{,}7 \\ 0 & 1 & -6.10^3 & -6{,}001.10^3 \\ 0 & 2{,}5 & 5 & 2{,}5 \end{array}\right) .(-2{,}5) \leftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -0{,}7 & 0 & 0{,}7 \\ 0 & 1 & -6.10^3 & -6{,}001.10^3 \\ 0 & 0 & 1{,}5005.10^4 & 1{,}5004.10^4 \end{array}\right)$$

In the **second** line of the **before last** table we have $-6{,}001.10^3.\ (-2{,}5) = 1{,}50025.10^4$. Here we have more than five significant digits and we must round off. We have two options:

1) Either remove the last digit and get $1{,}5002.10^4$.
2) Or roof it, in other words use $1{,}5003.10^4$.

Afterwards 2.5 is added to the result.

Let us assume that we use option 1). Then we will have: $1.5002.10^4+2,5=1,50045.10^4$ and the last digit will again be removed and ultimately we will have $1,5004.10^4$.

The last equation will have the form: $1,5005.10^4 x_3 = 1,5004.10^4$, where $x_3 = 0,99993$. At first this is good. Afterwards x_2 must be defined by the equation $x_2 - 6.10^3 x_3 = -6,001.10^3$, or $x_2 = -6,001.10^3 + 6.10^3.0,99993 = -1,5$.

Eventually from the first equation we calculate $x_1 = -0,35$.

We see, that the derived result (-0,35, -1,5, 0,99993) if far from the exact solution (0, -1, 1). Careful analysis shows that the loss of accuracy is due to the 'small' number 10^{-3} in the second line of the third transformation. The latter can be avoided when solving by using the Gauss method with a choice of a pivoting element.

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