Method of simple iteration for solving systems of linear algebraic equations (Jacobi method)

Let there be given a system of linear algebraic equations (SLAE)

(1) $Ax = b, \quad A = \left\{a_{ij}\right\}_{i,j=1}^{n}, \quad b = (b_1, b_2, \dots, b_n)^T$

and $x^* = (x_1^*, x_1^*, \dots, x_n^*)^T$ is the sought solution.

System (1) can be modified in the following equivalent form:

(2) $Ax = b \quad \leftrightarrow \quad \boxed{x = Cx + d}, \quad C = \left\{c_{ij}\right\}_{i,j=1}^{n},$

$c_{ii} = 0, \; c_{ij} = -a_{ij} / a_{ii}, \; d_i = b_i / a_{ii}, \; a_{ii} \neq 0 \;$ for $\; \forall i = \overline{1, n}$.

Using formula (2) it is possible to construct the iteration process:

(3) $x^{(k)} = Cx^{(k-1)} + d, \quad k = 1, 2, \dots$

or in extended form

$$\boxed{\left|\begin{array}{lllllll} x_1^{(k)} = & & c_{12}x_2^{(k-1)} + & \dots & + c_{1n}x_n^{(k-1)} & + d_1 \\ x_2^{(k)} = & c_{21}x_1^{(k-1)} + & & \dots & + c_{2n}x_n^{(k-1)} & + d_2 \\ \dots\dots\dots & & & & & \\ x_n^{(k)} = & c_{n1}x_1^{(k-1)} + & c_{n2}x_2^{(k-1)} + & \dots & & + d_n \end{array}\right. , \quad k = 1, 2, \dots}$$

The iteration process (3) results in the vector sequence

(4) $\boxed{x^{(0)}}, \; x^{(1)}, x^{(2)}, \dots, x^{(k)}, \dots$

From theory it is known, that in order to be sufficient for sequence (4) to converge towards the root x^* for every $x^{(0)}$ is for **<u>at least one norm of matrix *C* to be smaller than 1</u>**, i.e. **at least** one of the following inequalities should be true a), b) or c):

(5) a) $\sum_{j=1}^{n}\left|c_{ij}\right|<1, \quad i=1,2,\ldots,n$; b) $\sum_{i=1}^{n}\left|c_{ij}\right|<1, \quad j=1,2,\ldots,n$; c) $\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}^{2}<1$

For the closeness of the approximated solution $x^{(k)}$ to the exact solution x^{*} the following estimate is valid:

(6) $$\left\| x^{*}-x^{(k)}\right\| \le \|C\|^{k}\left(\left\|x^{(0)}\right\|+\frac{\|d\|}{1-\|C\|}\right)$$

Using formula (6) can be found the **minimal number of iterations** k needed to achieve some desired accuracy of ε. To do this it is sufficient to solve the following inequality regarding k:

(7) $$\|C\|^{k}\left(\left\|x^{(0)}\right\|+\frac{\|d\|}{1-\|C\|}\right)<\varepsilon \ .$$

NOTE!

The simple iteration method is especially suitable when the principle diagonal of matrix A is a dominating one, i.e.

$$|\, a_{ii}| >> \sum_{\substack{j=1 \\ i\neq j}}^{n}|\, a_{ij}| \ , \quad i=\overline{1,n}\,.$$

Algorithm

1. Constructing matrix C,
2. Convergence check using formulas (5),
3. Finding the **minimal** number of iterations in order to achieve the given accuracy ε using formula (7),
4. Implementing the resulting number of iterations (using formulas (3)).

Comments. Points 3. and 4. can be replaced by the so called stop criterion:

If $\left\|x^{(k)}-x^{(k-1)}\right\|<\varepsilon$, then $x^{*}=x^{(k)}$ with accuracy ε .

In coordinate form:

If $\left|x_i^{(k)}-x_i^{(k-1)}\right|<\varepsilon$, for $\forall i=\overline{1,n}$ then $x_i^{*}=x_i^{(k)}$ with accuracy ε.

Example. Perform five iterations using the Jacobi method for the system given below. Work with an intermediate accuracy of six digits after the decimal comma, and for initial guess choose the zero vector $x^{(0)} = (0,0,0)$.

$$\left| \begin{array}{llll} 4x_1 & -x_2 & & =2 \\ -x_1 & +4x_2 & -x_3 & =6 \\ & -x_2 & +4x_3 & =2 \end{array} \right.$$

Solution:

$\boxed{1.}$ $A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} \to C = \begin{pmatrix} 0 & 0{,}25 & 0 \\ 0{,}25 & 0 & 0{,}25 \\ 0 & 0{,}25 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \to d = \begin{pmatrix} 0{,}5 \\ 1{,}5 \\ 0{,}5 \end{pmatrix}.$

$\boxed{2.}$ Convergence check:

$$\sum_{j=1}^{3} |c_{1j}| = |0| + |0{,}25| + |0| = 0{,}25 < 1, \quad \sum_{j=1}^{3} |c_{2j}| = |0{,}25| + |0| + |0{,}25| = 0{,}5 < 1,$$

$$\sum_{j=1}^{3} |c_{3j}| = |0| + |0{,}25| + |0| = 0{,}25 < 1.$$

$\boxed{4.}$ Implementation of five iterations using formulas (3), which here have the following form:

$$\left| \begin{array}{lllll} x_1^{(k+1)} = & & 0{,}25x_2^{(k)} & & +0{,}5 \\ x_2^{(k+1)} = & 0{,}25x_1^{(k)} & & +0{,}25x_3^{(k)} & +1{,}5 \\ x_3^{(k+1)} = & & 0{,}25x_2^{(k)} & & +0{,}5 \end{array} \right. .$$

First iteration:

$$\left| \begin{array}{llllll} x_1^{(1)} = & & 0{,}25x_2^{(0)} & & +0{,}5 & = 0{,}25.0 \quad + 0{,}25 = 0{,}5 \\ x_2^{(1)} = & 0{,}25x_1^{(0)} & & +0{,}25x_3^{(0)} & +1{,}5 & = 0{,}25.0 + 0{,}25.0 + 1{,}5 = 1{,}5 \\ x_3^{(1)} = & & 0{,}25x_2^{(0)} & & +0{,}5 & = 0{,}25.0 + 0{,}5 = 0{,}5 \end{array} \right. \to \boxed{\begin{array}{l} x_1^{(1)} = 0{,}5 \\ x_2^{(1)} = 1{,}5 \\ x_3^{(1)} = 0{,}5 \end{array}}$$

Second iteration:

$$\left| \begin{array}{llllll} x_1^{(2)} = & & 0{,}25x_2^{(1)} & & +0{,}5 & = \quad 0{,}25.1{,}5 \quad + 0{,}5 = 0{,}875 \\ x_2^{(2)} = & 0{,}25x_1^{(1)} & & +0{,}25x_3^{(1)} & +1{,}5 & = 0{,}25.0{,}5 + 0{,}25.0{,}5 + 1{,}5 = 1{,}75 \\ x_3^{(2)} = & & 0{,}25x_2^{(1)} & & +0{,}5 & = \quad 0{,}25.1{,}5 + 0{,}5 = 0{,}875 \end{array} \right. \to \boxed{\begin{array}{l} x_1^{(2)} = 0{,}875 \\ x_2^{(2)} = 1{,}75 \\ x_3^{(2)} = 0{,}875 \end{array}}$$

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Do the remaining iterations on your own. The results have been entered in table 2.

Table 2

k \ $x^{(k)}$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	0,5	1,5	0,5
2	0,875	1,75	0,875
3	0,9375	1,9375	0,9375
4	0,984375	1,96875	0,984375
5	0,992188	1,992188	0,992188
…	...	...	...
x^*	1	2	1

How much iteration would be enough to calculate the same system with an accuracy $\varepsilon = 10^{-5}$ with second norm?

Solution:

Here we implement point [3.] of the algorithm. We have $x^{(0)} = (0,0,0)^T \rightarrow \left\| x^{(0)} \right\|_2 = 0$.

Also $d = (0{,}5, 1{,}5, 0{,}5)^T \rightarrow \|d\|_2 = 2{,}5$; $\|C\|_2 = 0{,}5$.

We solve the inequality with regard to k

$$\|C\|_2^k \left(\left\| x^{(0)} \right\|_2 + \frac{\|d\|_2}{1 - \|C\|_2} \right) < \varepsilon \quad \leftrightarrow$$

$$(0{,}5)^k \left(0 + \frac{2{,}5}{1 - 0{,}5} \right) < 10^{-5} \quad \leftrightarrow \quad (0{,}5)^k .5 < 10^{-5} \;\; | \;\; \lg \quad \leftrightarrow$$

$$k \lg 0{,}5 + \lg 5 < -5 \quad |: \lg 0{,}5 < 0 \quad \leftrightarrow \quad k > \frac{-5 - \lg 5}{\lg 0{,}5} \approx 18{,}93.$$

We choose $k = 19$.

Comment. If the diagonal elements of matrix A are only ones, then the method of consecutive approximations and Jacobi's method becomes one and the same thing.

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