EVLM

Spline iterpolation

Problem formulation

Let $y = f(x)$ be a function defined in the interval $[a,b]$ and a known table for the values $y_i = f(x_i)$ of the function in the points (nodes) $a \le x_0 < x_1 < x_2 < ... < x_n \le b$. Generally nodes are unequally distanced and we will mark the steps between them with $h_k = x_k - x_{k-1}$. Let the table be the following type:

x_i	x_0	x_1	...	x_i	...	x_n
y_i	y_0	y_1	...	y_i	...	y_n

The interpolation spline $S_k(f,x)$ of row k is a function with the following properties:

(1) $S_k(f,x)$ is a polynomial $f_i(x)$ with an exponent k in every sub-interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$.

(2) $S_k(f,x)$ interpolates the function, i.e. $S_k(f, x_i) = y_i$, $i = \overline{0,n}$,

(3) $S_k(f,x)$ and its derivatives up to row (k-1) are continuous throughout $[a,b]$.

When splines also satisfy other additional properties they are found in one way only. Most frequently used are splines from row k = 1, 2 and 3 which are respectively called linear, quadratic and cubic spline.

1. Linear spline

Here k = 1 i.e. in every sub-interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$, the spline $S_1(f,x)$ is a first degree polynomial (segment) which according to interpolation rule (2) connects points (x_{i-1}, y_{i-1}) and (x_i, y_i). Since between two points can exist only one segment, the linear spline is only one. When we interpolate the table for every interval we get the formulas given below which are used to calculate the coefficients of the linear spline using an output data table. The graphic of the spline is a broken line.

General formula of a linear spline	Spline coefficient
$S_1(f,x)=\begin{cases} f_1 = a_1 + b_1(x-x_0)\ , & x\in[x_0,x_1] \\ \dots & \\ f_i = a_i + b_i(x-x_{i-1})\ , & x\in[x_{i-1},x_i] \\ \dots & \\ f_n = a_n + b_n(x-x_{n-1})\ , & x\in[x_{n-1},x_n] \end{cases}$	$a_i = y_{i-1},$ $b_i = \dfrac{y_i - y_{i-1}}{h_i}, \quad i=\overline{1,n}$

Let it be so that we have calculated the coefficients of the linear spline using the formulas above and x' is a random point from $[a,b]$. To find the approximated value of the function $y=f(x)$, (i.e. $y(x')$) we first determine in which sub-interval $[x_{i-1},x_i]$ is located x', after which we substitute $x=x'$ in the respective row of $S_1(f,x)$.

2. Quadratic spline

When k = 2 according to property (1) the sought spline is a second degree polynomial (part of a parabola) in every sub-interval $[x_{i-1},x_i]$ and its coefficients are 3*n* in number. For their definition we use interpolation property (2), from which we get 2*n* equations and property (3) for the continuity of the first derivative, which results in *n*-1 more equations for all intermediate internal points $x_1, x_2, ..., x_{n-1}$. One condition remains unfulfilled, i.e. the quadratic spline isn’t the only one and can be determined by giving an additional condition. The formulas for calculation of coefficients for $S_2(x)$ are given in the table below where for definiteness the unfulfilled condition is given in the left side of the interval with $b_1=\gamma_1$. If $b_1=\gamma_1=0$ the spline is called **natural**. The procedure for the calculation of its coefficients is recurrent.

General formula of a quadratic spline	Spline coefficient
$S_2=\begin{cases} f_1 = a_1 + b_1(x-x_0)+c_1(x-x_0)^2, & x\in[x_0,x_1] \\ \dots & \\ f_i = a_i + b_i(x-x_{i-1})+c_i(x-x_{i-1})^2, & x\in[x_{i-1},x_i] \\ \dots & \\ f_n = a_n + b_n(x-x_{n-1})+c_n(x-x_{n-1})^2, & x\in[x_{n-1},x_n] \end{cases}$	$a_i = y_{i-1},$ $b_1=\gamma_1,$ $b_{i+1} = -b_i + 2\dfrac{y_i - y_{i-1}}{h_i},$ $c_i = \dfrac{b_{i+1}-b_i}{2h_i}, \quad i=\overline{1,n}$

3. Cubic spline

The third degree spline is analogically constructed and is defined by finding 4*n* coefficients under conditions (1)-(3). Since except interpolation property (2) here it is required that $S_3'(x)$ and $S_3''(x)$ are continuous and so 4*n*-2 conditions are formed and two conditions remain unfulfilled. Consequently the cubic spline is the only one when two additional conditions are set. In the table below they are marked with γ_1 and γ_2. To be more exact we are considering the conditions: $S_3''(a) = 2c_1 = l_1 = \gamma_1$ and $S_3''(b) = 2c_n + 6d_n h_n = l_{n+1} = \gamma_2$.

If $\gamma_1 = \gamma_2 = 0$ the spline is called **natural** cubic spline.

We will note that the representations of splines given here aren't the only ones. Except them there can be a number of other types of additional conditions, for example, $S_3'(a) = \gamma_1$ and $S_3'(b) = \gamma_2$ as well as different combinations of the conditions above, periodical conditions etc.

General formula of a cubic spline
$S_3 = \begin{cases} f_1 = a_1 + b_1(x-x_0) + c_1(x-x_0)^2 + d_1(x-x_0)^3, & x \in [x_0, x_1] \\ \dots \\ f_i = a_i + b_i(x-x_{i-1}) + c_i(x-x_{i-1})^2 + d_i(x-x_{i-1})^3, & x \in [x_{i-1}, x_i] \\ \dots \\ f_n = a_n + b_n(x-x_{n-1}) + c_n(x-x_{n-1})^2 + d_n(x-x_{n-1})^3, & x \in [x_{n-1}, x_n] \end{cases}$

Cubic spline coefficients
$a_i = y_{i-1},\quad b_i = \frac{y_i - y_{i-1}}{h_i} - \frac{h_i}{6}(l_{i+1} + 2l_i),\quad i = \overline{1,n}$ $c_i = \frac{l_i}{2},\quad d_i = \frac{l_{i+1} - l_i}{6h_i},\quad i = \overline{1,n-1}$ Here the auxiliary coefficients l_i are solutions to the following three diagonal system of linear algebraic equations with a predominating main diagonal: $\left\lvert \begin{array}{llll} l_1 & & & = \gamma_1 \\ h_1 l_1 & +2(h_1 + h_2) l_2 & +h_2 l_3 & = 6\left(\frac{y_2 - y_1}{h_2} - \frac{y_1 - y_0}{h_1}\right) \\ & \dots & & \\ h_{i-1} l_{i-1} & +2(h_{i-1} + h_i) l_i & +h_i l_{i+1} & = 6\left(\frac{y_i - y_{i-1}}{h_i} - \frac{y_{i-1} - y_{i-2}}{h_{i-1}}\right) \\ & & \dots & \\ & & l_{n+1} & = \gamma_2 \end{array}\right.$

Cubic spline coefficients in case of equally distanced nodes
$h = x_i - x_{i-1},\quad i = \overline{1,n}$ $a_i = y_{i-1},\quad b_i = \frac{y_i - y_{i-1}}{h} - \frac{h}{6}(l_{i+1} + 2l_i),\quad i = \overline{1,n}$ $c_i = \frac{l_i}{2},\quad d_i = \frac{l_{i+1} - l_i}{6h},\quad i = \overline{1,n-1}$

Auxiliary coefficients l_i are found from the system:

$$\left\lvert \begin{array}{llll} l_1 & & & = \gamma_1 \\ l_1 & +4l_2 & +l_3 & = 6\,\frac{y_0 - 2y_1 + y_2}{h^2} \\ & \dots\dots & & \\ l_{i-1} & +4l_i & +l_{i+1} & = 6\,\frac{y_i - 2y_{i-1} + y_{i-2}}{h^2} \\ & & \dots\dots & \\ & & l_{n+1} & = \gamma_2 \end{array}\right.$$

Example 1. The following table of values for function $y = f(x)$ is given:

x_i	3,0	4,5	7,0	9,0
y_i	2,5	1,0	2,5	0,5

Construct the stated spline and with its help find the approximate values of the function in points: $z_1 = 4$ and $z_2 = 5$:

a) linear spline

b) natural quadratic spline

c) natural cubic spline

Solution:

We calculate the steps: $h_1 =$ 4,5 - 3 = 1,5; $h_2 =$ 7 - 4,5 = 2,5; $h_3 =$ 9 – 7 = 2.

a) Using the formulas for S_1 we calculate consecutively coefficients a_i, b_i:

when $i = 1$ interval [3,0; 4,5]: $a_1 = y_0 = 2{,}5$, $b_1 = \dfrac{y_1 - y_0}{h_1} = \dfrac{1 - 2{,}5}{1{,}5} = -1$;

when $i = 2$ interval [4,5; 7]: $a_2 = y_1 = 1$, $b_2 = \dfrac{y_2 - y_1}{h_2} = \dfrac{2{,}5 - 1}{2{,}5} = 0{,}6$;

when $i = 3$ interval [7; 9]: $a_3 = y_2 = 2{,}5$, $b_3 = \dfrac{y_3 - y_2}{h_3} = \dfrac{0{,}5 - 2{,}5}{2} = -1$.

We enter the coefficients and get the following table:

i	a_i	b_i	Linear spline $S_1(f, x)$ from example 1a)
1	2,5	-1,0	$f_1 = 2{,}5 - (x - 3)$ when $x \in [3; 4{,}5]$
2	1,0	0,6	$f_2 = 1 + 0{,}6(x - 4{,}5)$ when $x \in [4{,}5; 7]$
3	2,5	-1,0	$f_3 = 2{,}5 - (x - 7)$ when $x \in [7; 9]$

To calculate the approximate value of the function in point $z_1 = 4$ with the help of the spline we determine that it is located in the first interval and will approximate using the formula for f_1. Then

$f(4) \approx f_1(4) = a_1 + b_1(z_1 - x_0) = 2{,}5 + (-1)(4 - 3) = 1{,}5$.

Analogically $f(5) \approx f_2(5) = a_2 + b_2(z_2 - x_1) = 1 + 0{,}6(5 - 4{,}5) = 1{,}3$.

The graphic of the spline is shown in fig. 1 – S_1 .

It is easy to ascertain that the spline is correctly determined. In the case of a linear spline to do this we need only to check if it goes through points y_i, $i = 1,2,3,4$ and that it is continuous. Indeed $S_1(3) = f_1(3) = 2{,}5$, $S_1(4{,}5) = f_2(4{,}5) = 1$ and $S_1(7) = f_2(7) = 2{,}5$, also:

$f_1(4{,}5) = 2{,}5 + (-1).(4{,}5 - 3) = 2{,}5 - 1{,}5 = 1 = f_2(4{,}5)$, $f_2(7) = 1 + (0{,}6).(2{,}5) = 1 + 1{,}5 = 2{,}5 = f_3(7)$ and $f_3(9) = 2{,}5 + (-1).(9 - 7) = 2{,}5 - 2 = 0{,}5$.

b) In the case of a natural quadratic spline s_2 the calculations are carried out using recurrent formulas from the respective table. For a_i, b_i, c_i when $i = 1, 2, 3$ and $\gamma_1 = 0$:

i	a_i	b_i	c_i	Quadratic spline $S_2(f,x)$ from example 1b)	
1	2,5	0,0	-0,6667	$f_1 = 2{,}5 - 0{,}6667(x-3)^2$	when $x \in [3; 4{,}5]$
2	1,0	-2,0	1,0400	$f_2 = 1 - 2(x - 4{,}5) + 1{,}04(x - 4{,}5)^2$	when $x \in [4{,}5; 7]$
3	2,5	3,2	-2,1000	$f_3 = 2{,}5 + 3{,}2(x-7) - 2{,}1(x-7)^2$	when $x \in [7; 9]$
4	-	-5,2	-	-	

The approximation of the function in the point $z_1 = 4$ is found from

$$f(4) \approx f_1 = a_1 + b_1(z_1 - x_0) + c_1(z_1 - x_0)^2 = 2{,}5 - 0{,}6667(4-3)^2 = 1{,}8333.$$

Analogically for the other point $z_2 = 5$ we get

$$f(5) \approx f_2 = a_2 + b_2(z_2 - x_1) + c_2(z_2 - x_1)^2 = 1 - 2(5 - 4{,}5) + 1{,}04(5 - 4{,}5)^2 = 0{,}26 \ .$$

c) Now let us also construct the cubic spline S_3. For this we form the linear system for the auxiliary coefficients l_i where $i = 1, 2, 3, 4$ and $\gamma_1 = \gamma_2 = 0$:

$$\left| \begin{array}{llllll} l_1 & & & & = 0 \\ h_1 l_1 & +2(h_1+h_2)l_2 & +h_2 l_3 & & = 6\left(\dfrac{y_2-y_1}{h_2} - \dfrac{y_1-y_0}{h_1}\right) \\ & h_2 l_2 & +2(h_2+h_3)l_3 & +h_3 l_4 & = 6\left(\dfrac{y_3-y_2}{h_3} - \dfrac{y_2-y_1}{h_2}\right) \\ & & & l_4 & = 0 \end{array} \right. .$$

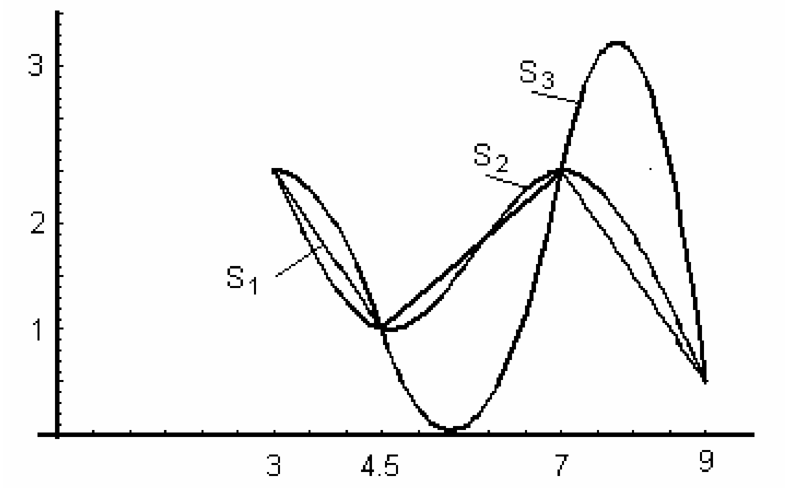

Fig. 1. The graphics of the resulting splines for problem 1a), 1b) and 1c).

By substituting the concrete data we get the following system:

$$\begin{vmatrix} 8l_2 + 2{,}5l_3 = 9{,}6 \\ 2{,}5l_2 + 9l_3 = -9{,}6 \end{vmatrix}.$$

Solving this system for the auxiliary coefficients we find that:

$l_1 = 0,\ l_2 = 1{,}6791,\ l_3 = -1{,}5331,\ l_4 = 0$.

After substituting in the formulas we get the following table for the coefficients:

i	l_i	a_i	b_i	c_i	d_i
1	0,0000	2,5	-1,4198	0,0000	0,1866
2	1,6791	1,0	-0,1605	0,8395	-0,2141
3	-1,5331	2,5	0,0221	-0,7666	0,1278
4	0,0000	-	-	0,0000	-

With the help of these coefficients the sought cubic spline is written down in the following way:

Table for cubic spline $S_3(f,x)$ from example 1c)	
$f_1 = 2{,}5 - 1{,}4198(x-3) + 0{,}1866(x-3)^3$	when $x \in [3;\ 4{,}5]$
$f_2 = 1 - 0{,}1605(x-4{,}5) + 0{,}8395(x-4{,}5)^2 - 0{,}2141(x-4{,}5)^3$	when $x \in [4{,}5;7]$
$f_3 = 2{,}5 + 0{,}0221(x-7) - 0{,}7666(x-7)^2 + 0{,}1278(x-7)^3$	when $x \in [7;9]$

As in the previous cases in order to get approximated values in points $z_1 = 4$ and $z_2 = 5$ with the help of the cubic spline we make substitutions in the general formulas depending on the interval in which the point is located. For the first point we have

$$f(4) \approx f_1(4) = a_1 + b_1(z_1 - x_0) + c_1(z_1 - x_0)^2 + d_1(z_1 - x_0)^3 =$$

$$2{,}5 - 1{,}4198.(4-3) + 0.(4-3)^2 + 0{,}1866.3^3 = 1{,}2668 .$$

For the second point -

$$f(5) \approx f_2(5) = a_2 + b_2(z_2 - x_1) + c_2(z_2 - x_1)^2 + d_2(z_2 - x_1)^3 =$$

$$1 - 0{,}1605.(5-4{,}5) + 0{,}8395.(5-4{,}5)^2 - 0{,}2141.(5-4{,}5)^3 = 1{,}1029 .$$

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