Function approximation by Lagrange's interpolation polynomial

Problem formulation

Let M and P are two sets of functions defined in one and the same domain such that $P \subset M$. Set M also contains functions which are "difficult" to work with (value finding, differentiation, integration etc.) while P contains functions that are "easy" to work with (continuous, polynomials). For each function $f(x)$ from the set M we are looking for that function $p(x)$ from the set P which is "closest" to $f(x)$. As $p(x)$ is "close" to $f(x)$ we can hope that the results of operations with $p(x)$ will be "close" to the results of the same operations with $f(x)$ and are interchangeable. By specifying the concept of "closeness" we get the different ways for approximation of functions.

Interpolation

We consider functions $f(x)$ given in a table:

x_i	x_0	x_1	$\dots$	x_n
$y_i = f(x_i)$	y_0	y_1	$\dots$	y_n

And the set $P \equiv \prod_n(x)$- polynomials of ***n*** -th degree of the variable x. If $x_i \neq x_j$ for $i \neq j$ exists a single polynomial of ***n*** –th degree which assumes in x_i a value of y_i. This polynomial is called **Lagrange's interpolation polynomial** and has the following form:

$$L_n(x) = \sum_{i=0}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

In this case by "closeness" between $f(x)$ and $L_n(x)$ we mean:

$L_n(x_i) = f(x_i) = y_i, \quad i = 0,...,n$ - the values in $(n+1)$ coincide in different points.

If: $x_0 < x_1 <...< x_n$ anf $x \in [x_0, x_n]$ the process is called **interpolation** of $f(x)$ with $L_n(x)$ and if $x \notin [x_0, x_n]$ the process is called **extrapolation** of $f(x)$ with $L_n(x)$. If $f(x)$ has a continuous $(n+1)$ –th derivative the absolute error in arbitrary point x is evaluated in the following way:

$$|R_n(x)| = |f(x) - L_n(x)| \le \frac{M_{n+1}}{(n+1)!}|(x-x_0)...(x-x_n)|,$$

where $M_{n+1} = \max\limits_{[p,q]}\left|f^{(n+1)}(x)\right|$ and $[p,q] \equiv [x_0, x_n]$ for interpolation;

$[p,q] \equiv [\min(x, x_0), \max(x, x_n)]$ for extrapolation.

Example 1. Using the values of $f(x) = \sqrt{x}$ for the values of an argument 100, 121, 144 find the approximated value of $\sqrt{115}$ (without using the root extraction) and evaluate the error of approximation.

Solution:

Using the table

x	100	121	144
$y = \sqrt{x}$	10	11	12

we construct a second degree interpolation polynomial:

$$L_2(x) = 10\frac{(x-121)(x-144)}{(100-121)(100-144)} + 11\frac{(x-100)(x-144)}{(121-100)(121-144)} + 12\frac{(x-100)(x-121)}{(144-100)(144-121)} =$$

$$= \frac{10}{21.44}(x-121)(x-144) - \frac{11}{21.23}(x-100)(x-144) + \frac{12}{44.23}(x-100)(x-121)$$

Then for $x = 115$ we get:

$$L_2(115) = 10\frac{(115-121)(115-144)}{21.144} + 11\frac{(115-100)(115-144)}{21.23} + 12\frac{(115-100)(115-121)}{44.23}$$

or $L_2(115) = 10{,}7227555...$

To evaluate the error (accuracy) of the result, we consecutively find:

$$y = \sqrt{x} = x^{\frac{1}{2}}\ ;\ y' = \frac{1}{2}x^{-\frac{1}{2}}\ ;\ y'' = -\frac{1}{4}x^{-\frac{3}{2}}\ ;\ y''' = \frac{3}{8}x^{-\frac{5}{2}}\ ;$$

$$M_3 = \max_{[100,144]}\left|\frac{3}{8}x^{-\frac{5}{2}}\right| = \frac{3}{8\sqrt{100^5}} = \frac{3}{8.10^5}\ ;$$

$$|R_2(115)| \le \tfrac{1}{3!}\left(\tfrac{3}{8.10^5}\right).|(115-100)(115-121)(115-144)| = 0{,}00163125...$$

i.e. the error isn't more than 0,002.

Note. If we want to (or have to) do the intermediate calculations by rounding off the results it is best to begin with error evaluation and then proceed rounding off so that the error does not interfere with the accuracy of the result. In the example above it is sufficient to work with accuracy 10^{-4}, i.e. rounding off up to the fourth digit after the decimal sign, as the error of the method is in the third digit.

Solution 2. What does the accuracy have to be to tabulate (calculate) the values $\ln(100)$, $\ln(101)$, $\ln(102)$ and $\ln(103)$ so that it does not influence the error for the interpolation of $\ln(100{,}5)$?

Solution:

For the four values $\ln(100)$, $\ln(101)$, $\ln(102)$ and $\ln(103)$ we can construct a third degree interpolation polynomial. Then the interpolation error for $\ln(100{,}5)$ can be evaluated in this way:

$$|R_3(100{,}5)| \le \frac{M_4}{4!}|(100{,}5-100).(100{,}5-101).(100{,}5-102).(100{,}5-103)|.$$

To determine M_4 we find consecutively:

$$y = \ln x, \quad y' = \frac{1}{x} = x^{-1}, \quad y'' = -x^{-2},$$

$$y''' = 2x^{-3}, \quad y^{IV} = -6x^{-4}, \quad M_4 = \max_{[100,144]} \left| \frac{-6}{x^4} \right| = \frac{6}{100^4} = 6.10^{-8}.$$

After substituting in R_3 we get: $R_3(100{,}5) \le 2{,}3.10^{-9}$ i.e. the values of $\ln(100)$, $\ln(101)$, $\ln(102)$ and $\ln(103)$ have to be tabulated at least up to the 9th digit after the decimal sign.

Example 3. The table is given:

x	10	15	17	20
$y = f(x)$	3	7	11	17

Using Lagrange's interpolation polynomial find the approximated solution of the equation $y = f(x) = 10$.

Solution:

We can solve the problem in two different ways.

I: Let us construct $L_3(x)$ for the table above and then solve (with exacts or approximations) the equation $L_3(x) = 10$.

II: Lets "reverse" the table and switch the places of the first and the second row. This way we get a table for the values of the reverse function $x = x(y)$. Using this table we construct an interpolation polynomial and its value when $y = 10$ is the solution to the problem. This method can only be used if $y_i \neq y_j$ за $i \neq j$ (Why?) and it is called <u>backward interpolation</u>. Let us solve the problem as in this case we don't have to solve a polynomial equation.

We construct Lagrange's interpolation polynomial using the table:

y	3	7	11	17
$x = x(y)$	10	15	17	20

$$L_3(x) = 10\frac{(y-7)(y-11)(y-17)}{(3-7)(3-11)(3-17)} + 15\frac{(y-3)(y-11)(y-17)}{(7-3)(7-11)(7-17)} +$$

$$+17\frac{(y-3)(y-7)(y-17)}{(11-3)(11-7)(11-17)} + 20\frac{(y-3)(y-7)(y-11)}{(17-3)(17-7)(17-11)} =$$

$$= -\frac{5}{224}(y-7)(y-11)(y-17) + \frac{3}{32}(y-3)(y-11)(y-17) -$$

$$= -\frac{17}{192}(y-3)(y-7)(y-17) + \frac{1}{42}(y-3)(y-7)(y-11).$$

Its value when $y = 10$ is $L_3(10) = 16{,}640625$ which we accept to be the solution of the problem.

Example 4. The function $y = \sin(x)$ is tabulated with step 1^0 in the interval $[0, \pi]$ ($1^0 = \pi/180$). Evaluate the error in the case of linear interpolation of this function for $\forall x \in [0, \pi]$.

Solution:

Let $x \in [x_k, x_{k+1}]$ is the k –th interval of the division of $[0, \pi]$ with step $h = 1^0 = 0{,}01745329\ldots$ Then for the error of the linear interpolation we will have

$$|R_1(x)| \le \frac{M_2}{2!}|(x - x_k)(x - x_{k+1})|.$$

For the function $y = \sin(x)$ we have $y'' = -\sin x \Rightarrow M_2 = 1$, $\forall x \in [0, \pi]$. If we replace $t = x - x_k$, $x - x_{k+1} = x - (x_k + h) = x - x_k - h = t - h$ we have $|R_1(x)| \le \frac{1}{2}|t(t-h)|$. We will find the biggest absolute value of the function $\varphi(t) = t(t-h)$ when $t \in [0, h]$ (as while interpolating $x \in [x_k, x_{k+1}]$).

$$\varphi'(t) = 2t - h, \qquad \varphi'(h/2) = 0, \qquad \varphi(h/2) = -h^2/4,$$

$$\varphi(0) = \varphi(h) = 0 \Rightarrow \max_{[0,h]}|\varphi(t)| = \frac{h^2}{4}$$

Finally $|R_1| \le \frac{1}{2}\frac{h^2}{4} = \frac{h^2}{8} = 0{,}000038\ldots \approx 0{,}00004$.

Example 5. Find those nodes in $[1, 2]$ which minimize the error for the interpolation of a random function with a third degree polynomial. Evaluate this error for $f(x) = \sqrt{x}$.

Solution:

It is known that the nodes minimizing the error for a random interval $[a,b]$ are: $t_k^* = \frac{b-a}{2} x_k^* + \frac{b+a}{2}$, where $x_k^* = \cos\frac{(2k+1)\pi}{2(n+1)}$, $k = 0 \div n$ are the zeros of Chebyshev’s polynomial of the first kind and the evaluation for the error in this case is:

$$|R_n| \le \frac{(b-a)^{n+1}}{2^{2n+1}} \cdot \frac{M_{n+1}}{(n+1)!}.$$

$$n = 3 \Rightarrow x_0^* = \cos\frac{\pi}{8},\ x_1^* = \cos\frac{3\pi}{8},\ x_2^* = \cos\frac{5\pi}{8},\ x_3^* = \cos\frac{7\pi}{8}$$

$$t_0^* = \frac{1}{2}\cos\frac{\pi}{8} + \frac{3}{2} = 1{,}961939765...,\ t_1^* = \frac{1}{2}\cos\frac{3\pi}{8} + \frac{3}{2} = 1{,}691341715...,$$

$$t_2^* = \frac{1}{2}\cos\frac{5\pi}{8} + \frac{3}{2} = 1{,}308658285... = 3 - t_1^*,\ t_3^* =$$

$$= \frac{1}{2}\cos\frac{7\pi}{8} + \frac{3}{2} = 1{,}038060235... = 3 - t_0^*.$$

For the function $f(x) = \sqrt{x}$ we have $y^{IV} = -\frac{15}{16} x^{-\frac{7}{2}}$, then

$$M_4 = \max_{[1,2]}\left|-\frac{15}{16} x^{-\frac{7}{2}}\right| = \frac{15}{16},\quad |R_3| \le \frac{(2-1)^4}{2^7} \cdot \frac{M_4}{4!} = \frac{15/16}{2^7.24} = \frac{5}{2^{14}} = 0{,}000305...$$

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