Introduction to numerical solving of algebraic equations and systems of equations

We are considering the problems:

1. Finding out the real roots of one equation
2. Solving systems of linear algebraic equations
3. Solving systems of nonlinear algebraic equations

At numerical calculations we often have to find the roots of equation with single unknown. For example the equation

$$x^2 - x - 2 = 0$$

has two different real roots $x_1 = -1$ и $x_2 = 2$, which are defined by the formulas for solving a square equation. Much more difficult is to solve an arbitrary transcendent equation, for example the equation:

$$2x\left(x-3\right)+tg\left(x+1\right)-3=0. \qquad (1)$$

Relatively easy way to discover the approximate location of the real roots is to draw a graph of the function, in our case it is the function $f(x)=2x\left(x-3\right)+tg\left(x+1\right)-3$. With the aid of *Mathematica* package for example we could obtain the graph in the interval [-5, 5] in the following manner:

```
f[x_]:=2x(x-3)+Tan[x+1.]-3     (*Defining the formula  of the function*)
g=Plot[f[x], {x,-5,5}]         (* Obtaining two-dimensional graph of f(x)*)
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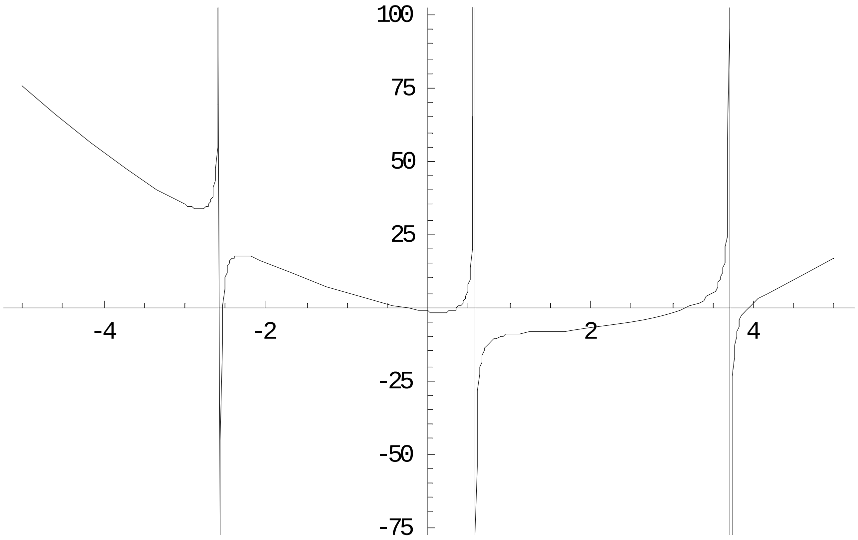
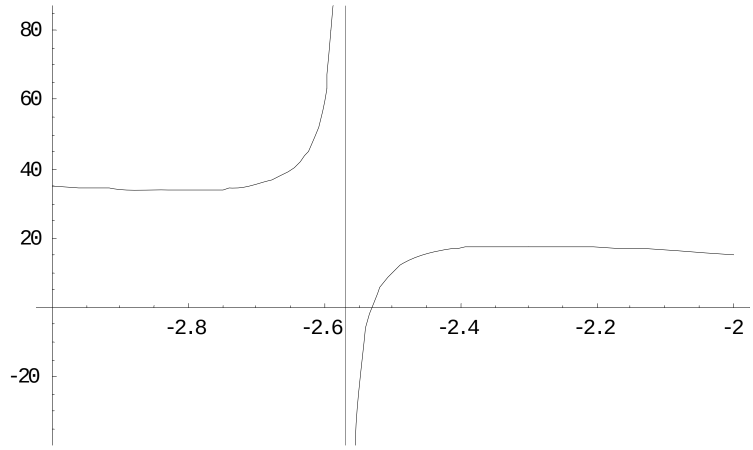


Fig. 1. Graph of f(x) in the interval [-5,5].

It's obvious that $f(x)=0$ has five real roots in the intervals [-3, -2], [-1, 0], [0, 1] respectively and two roots in [3, 4]. But as the function is discontinuous in order to localize the roots more precisely we draw the graphs closer to them. For example for the smallest root in the interval [-3, -2] we see that the root can be localized with the function graph in the interval [-2.55, -2.5]. The values of $f(x)$ in the periphery of this interval are with opposite signs, -22.77 and +10.40 correspondingly and consequently out of that discontinuity at some point it nullifies:

```
g1=Plot[f[x], {x,-3,-2}]
```




```
f[-2.55]
f[-2.5]
-22.7735
10.3986
```

If we select proper intervals the other roots can be localized by analogy.

When we localize particular root, we need to apply any numerical method to calculate it more precisely.

In the common case the problem of solving algebraic equation

$$f(x)=0 \tag{2}$$

is reduced to the following three sub problems:

1) To explore if there are roots (zeros) of the equation, to determine the number of real and complex roots and their multiplicity – simple, double etc.

2) To discover intervals or regions containing single root. That is called localization of zeros.

3) To calculate the roots with a given accuracy.

Solving of the sub problems above is usually by analytical or graphical way and the determination of the roots – through numerical method - manually or computer aided. The value of the roots rarely can be obtained exactly. Usually it is approximated due to rounding errors, error of the method etc.

Another important problem is solving systems of equations. They can be linear and nonlinear systems.

The common form of a linear algebraic system of *n* equations with *n* unknown values can be written in the form

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{vmatrix} \quad (3)$$

where x_1, x_2, ... ,x_n are the unknown values and a_{ij}, b_i $(i = \overline{1,n},\ j = \overline{1,n})$ are given real coefficients.

Problem (3) can be expressed in vector form

$$A\vec{x} = \vec{b} \quad (4)$$

where we use the annotations:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

From the linear algebra we know that the necessary and sufficient condition for existence and uniqueness of the solution $\vec{x} = (x_1, \ldots, x_n)$ is:

$$\det A \neq 0. \quad (5)$$

Example of a system of linear equations is the system:

$$\begin{vmatrix} 5x_1 + 3x_2 - 4x_3 = 0 \\ 2x_1 - 5x_2 + x_3 = 2 \\ 4x_1 - 7x_2 - 5x_3 = -3 \end{vmatrix}. \quad (6)$$

It can be solved through the Cramer's formulae that use sub-determinants but that method is inefficient at $n \geq 3$. That's why for the efficient solving of systems of linear algebraic equations are created many numerical methods and further we will examine the most widely used.

With the help of the system *Mathematica* we also can solve that system but here we only will find the determinant of the system using the code:

```
A={{5,3,-4},{2,-5,1},{4,-7,-5}} (* setting of matrix A *)
Det[A]                          (* calculation of the determinant*)
{{5,3,-4},{2,-5,1},{4,-7,-5}}
 178
```

Since the determinant = 178 ≠0, then this system has a single solution.

Finally we'll note the important for the applications case of systems of nonlinear equations. Their general form is

$$\left| \begin{array}{l} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ \qquad \vdots \\ f_n(x_1, x_2, ..., x_n) = 0 \end{array} \right. \qquad (7)$$

or by the vector representation

$$\vec{F}(\vec{x}) = \vec{0}\,. \qquad (8)$$

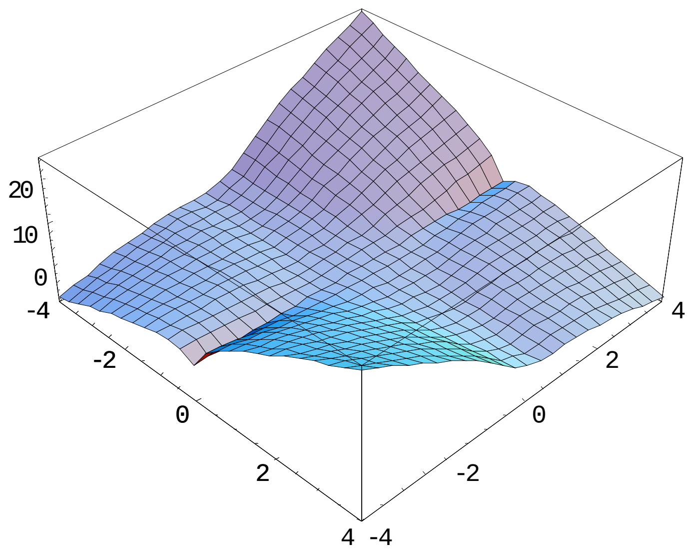
Solving of the system (8) is much more complex problem then the linear case (3) as a particular preliminary exploration is necessary by analogy with the general problem of one equation with single unknown. There are no theorems for the existing of solutions in the common case.

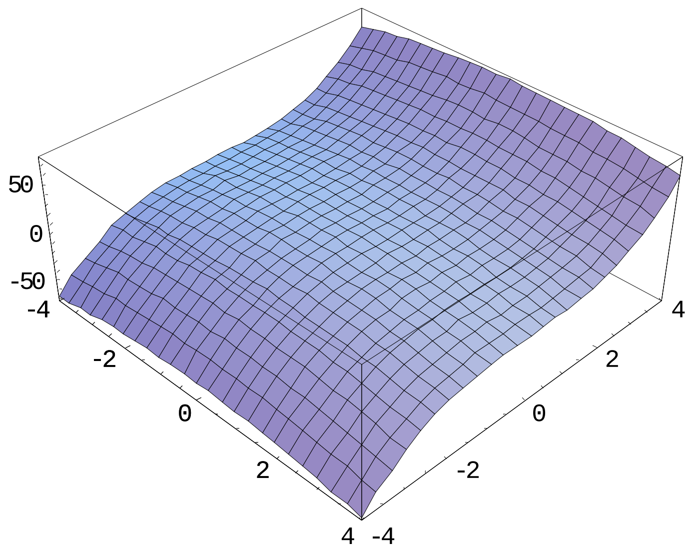
Nonlinear system of equations is for example the system:

$$\left| \begin{array}{l} \ln(1 + x^2 + y^4) = x \\ \sqrt{x^2 + y} + (2 - x + y^3)^2 = 0 \end{array} \right.$$

With the aid of the system *Mathematica* we can build graphs with the code:

```
f1[x_, y_] := Log[1. + x^2 * y^4] - x y
f2[x_, y_] := 2. - x^2 + y^3
r1 = Plot3D[f1[x, y], {x, -4, 4}, {y, -4, 4}, ViewPoint -> {2., -2, 2}]
r2 = Plot3D[f2[x, y], {x, -4, 4}, {y, -4, 4}, ViewPoint -> {2., -2, 2}]
```





The intersection of the two graphs shows where the functions intersect. By the common points we search these for which the values of both functions are zero. To determine the roots we use particular numerical methods.

Here is the intersection of two graphs from which we can see that the continuous curve in domain and its section with the plane *Oxy* probably has common points – the roots we search:

```
Show[r1,r2, ViewPoint->{2., -2, 2}]
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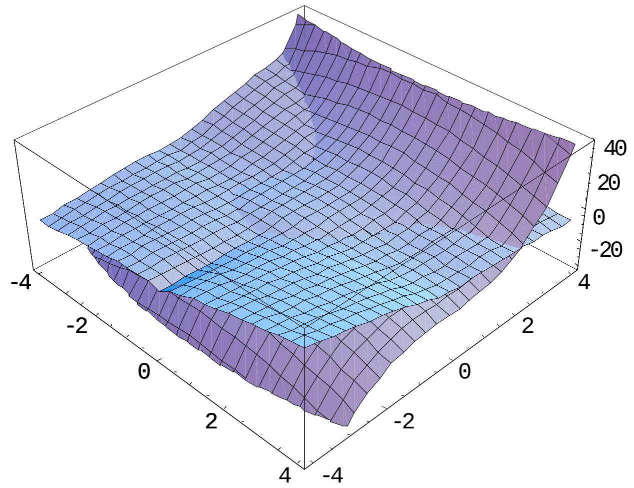

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