Absolute and relative error

The absolute error is a measure of approximation of one number to another. The approximate value in principle is a rational quantity and could replace the exact number in further calculations.

Let $\bar{x}$ be an exact number, and $\tilde{x}$ is its approximation. The absolute error of $\tilde{x}$ to $\bar{x}$ is defined as $\alpha(\tilde{x})$ and is expressed with the formula:

$$\alpha(\tilde{x}) \geq |\bar{x} - \tilde{x}| . \qquad (1)$$

Note that the absolute value is not the exact but some upper limit of the absolute difference between $\bar{x}$ and $\tilde{x}$. This allows us to take for $\alpha(\tilde{x})$ any number which is greater than this absolute distance. Often one take convenient values for $\alpha(\tilde{x})$, such as 0.0001, 0.000001, 0.0005 etc. It must be added that the absolute error depends on the physical measure: metre, kilogramme,

If we solve the inequality (1) in respect of $\bar{x}$ we'll receive

$$\tilde{x} - \alpha(\tilde{x}) \leq \bar{x} \leq \tilde{x} + \alpha(\tilde{x}) .$$

This means that the exact number is somewhere in the interval $[\tilde{x} - \alpha(\tilde{x}), \tilde{x} + \alpha(\tilde{x})]$. Often the exact number $\bar{x}$ could be unknown but certain approximate $\tilde{x}$ and absolute error $\alpha(\tilde{x})$ are known. In this case with tolerance $\pm\alpha(\tilde{x})$ for approximation of $\bar{x}$ may be applied every number of the interval

$$[\tilde{x} - \alpha(\tilde{x}),\ \tilde{x} + \alpha(\tilde{x})].$$

The relative error is another criterion for evaluation of the closeness between some exact number $\bar{x}$ and its approximate number $\tilde{x}$. The relative error is noted by $\Delta(\tilde{x})$ and is calculated by the inequality:

$$\Delta(\tilde{x}) \geq \frac{\alpha(\tilde{x})}{|\tilde{x}|} \qquad (2)$$

or

$$\Delta(\tilde{x}) \geq \frac{|\bar{x} - \tilde{x}|}{|\tilde{x}|} \ . \qquad (3)$$

Usually $\Delta(\tilde{x})$ is expressed in percents.

Example. Find the absolute and relative errors of the number $x = 2/7$ and two of its approximations $\tilde{x}_1 = 0,286$ and $\tilde{x}_2 = 0,2857$.

Solution. We have $x = \frac{2}{7} = 0,285714285...$.

For the absolute errors we calculate:

$$\left|x - \tilde{x}_1\right| = \left|-0,00028571...\right| \le \alpha(\tilde{x}_1)\,, \quad \left|x - \tilde{x}_2\right| = \left|0,00001428...\right| \le \alpha(\tilde{x}_2)\,.$$

We can choose for example: $\alpha(\tilde{x}_1) = 0,0029$ or $\alpha(\tilde{x}_1) = 0,003$.

Respectively for $\tilde{x}_2$: $\alpha(\tilde{x}_2) = 0,0002$, $\alpha(\tilde{x}_2) = 0,0005$ or $\alpha(\tilde{x}_2) = 0,001$.

For the relative errors we obtain:

$$\Delta(\tilde{x}_1) = \frac{\alpha(\tilde{x}_1)}{\left|\tilde{x}_1\right|} = \frac{0,003}{0,286} = 0,001048...,\ \Delta(\tilde{x}_1) = 0,002 \text{, also as } \Delta(\tilde{x}_1) = 0,2\%\,.$$

$$\Delta(\tilde{x}_2) = \frac{\alpha(\tilde{x}_2)}{\left|\tilde{x}_2\right|} = \frac{0,0005}{0,2857} = 0,000070...,\ \Delta(\tilde{x}_2) = 0,0001 \text{ or } \Delta(\tilde{x}_2) = 0,01\%\,.$$