7^{th} Lecture

Local, Constrained and Global Extrema for Functions of Two Variables.

Local Extrema.

Local extrema for functions of several variables are defined completely in the same way as for functions of one variable.

Let f be a function of n variables and let $A \in D(f) \subset E_n$. It is said, that the value $f(A)$ is the **local maximum (minimum)** of f, if there exists such a neighbourhood $N_\varepsilon(A)$, that for each $X \in N_\varepsilon(A)$ is $f(X) \leq f(A$ $(f(X) \geq f(A))$. If for all $X \in N_\varepsilon(A)$, $X \neq A$, is $f(X) < f(A)$ $(f(X) > f(A))$, $f(A)$ is said to be the **strict local maximum (minimum)**. The point A is then called the **point of (strict) local maximum or minimum**.

Example 1. Show, that the function $f(x,y) = (x-1)^2 + (y-2)^2 - 1$ has the strict local minimum -1 at the point $[1,2]$ and no local maximum.

In what follows we want to learn how to find local extrema for functions of two variables.

Let f be a function of two variables. A point $A \in D(f)$ is said to be a **critical point** of the function, if $f'_x(A)$ and $f'_y(A)$ vanish or do not exist.

It can be proved, that a function f possess local extrema only at critical points.

Example 2. Find all critical points and local extrema of functions

1. $f(x,y) = |x| + |y|$,
2. $f(x,y) = -\sqrt{x^2+y^2}$,
3. $f(x,y) = 4 - x^2 - y^2$.

If $f'_x(A) = f'_y(A) = 0$, but $f(A)$ is not any local extremum, then the point A is called the **saddle point** of f.

Example 3. Show, that the function $f(x,y) = x^2 - y^2$ has no local extremum and the point $[0,0]$ is its saddle point.

It follows, that really, the fact that $f'_x(A) = f'_y(A) = 0$ does not itself guarantee that the value $f(A)$ is a local extremum. However, if f and its first and second derivatives are continuous in an $N_\varepsilon(A)$, there is the **second derivative test**, that may verify the behaviour of the function f at the point A:

Let a point $A = [x_0, y_0]$ be a critical point of a function $f(x, y)$, having continuous first and second partial derivatives at a neighbourhood $N_\varepsilon(A)$. Let

$$D(x_0, y_0) = f''_{xx}(x_0, y_0).f''_{yy}(x_0, y_0) - (f''_{xy}(x_0, y_0))^2$$

Then
a) f has at A a strict local maximum, if $D(x_0, y_0) > 0$, and $f''_{xx}(x_0, y_0) < 0$,
b) f has at A a strict local minimum, if $D(x_0, y_0) > 0$, and $f''_{xx}(x_0, y_0) > 0$,
c) f has not at A any strict local extremum, if $D(x_0, y_0) < 0$,
d) the test fails, if $D(x_0, y_0) = 0$.

Example 4. Find all local extrema and saddle points of functions

1. $f(x, y) = 3x^2 - 2xy + y^2 - 8y$,
2. $f(x, y) = x^3 + 3xy^2 - 15x - 12y$,
3. $f(x, y) = 4xy - x^4 - y^4$,
4. $f(x, y) = -y^3$,
5. $f(x, y) = x \sin y$,
6. $f(x, y) = x^4 + y^4$.

Constrained Extrema.

In problems involving the determination of extrema of functions of two variables we often encounter the so called **constrained (conditional) extrema**. Let there be given a function f and a set (for example a curve) $M \subset D(f)$. The problem is to find a point $A \in M$ such that the value $f(A)$ is the greatest or the least, compared with values of f at the points of the set M, lying "close" to the point A. A point A of this kind is called a **point of constrained extremum**.

It means: Let f be a function of two variables and let a set $M \subset D(f)$. Then a point $A \in M$ is called the **point of constrained local maximum (minimum)**, if there exists such a neighbourhood $N_\varepsilon(A)$, that for each $X \in M \cap N_\varepsilon(A)$ it is valid $f(X) \leq f(A)$ $(f(X) \geq f(A))$.

The set M is mostly given as a set of points from $D(f)$ satisfying an equation (a condition):

$$M = \{[x, y] \in D(f) : g(x, y) = 0\}$$

The equation $g(x, y) = 0$ is called the **constraint**. It is easier to solve this problem in the case, that from the equation $g(x, y) = 0$ it is possible to express the variable x or y as a function of the other variable. If, for example, from the constraint we obtain $y = \varphi(x)$, then when we substitute this expression for y to the the original function f we have a function of one variable

$$F(x) = f(x, \varphi(x)).$$

In this way instead of constrained local extrema of the function of two variables f, we look for local extrema of the function of one variable, F. Similarly, if $x = \psi(y)$.

Example 5. Find constrained local extrema of the function $f(x, y) = x^2 + y^2 + 1$, if the constraint is $x + y - 1 = 0$. Explain the geometric meaning of the problem.

Example 6. Find dimensions of the rectangle with the perimeter 10, having the greatest area.

However, if the constraint $g(x, y) = 0$ is too complicated to express one of the variables in terms of the other, then another method can be used, namely the **Method of Lagrange multiplier(s)**. To determine the points at which the function f attains constrained local extrema, we form an auxiliary function

$$\Phi(x, y) = f(x, y) + \lambda g(x, y),$$

where λ is an arbitrary constant, called the **Lagrange multiplier**. It is clear, that the function Φ is defined on the set M and moreover

$$\Phi(x, y) = f(x, y) \quad \text{for each } [x, y] \in M.$$

It can be easily proved: If a point $A = [x_0, y_0] \in M$ is a point of local extremum for the function Φ, then A is a point of constrained local extremum for the function f, subject to the constraint $g(x, y) = 0$.

Opposite of the latter proposition is not valid. It can happen, that we can not find all constrained local extrema with the aid of this method.

Applying the method of Lagrange Multiplier, first we create the following system of three equations in three unknowns $x,\ y,\ \lambda$

$$\begin{aligned} \Phi'_x(x, y) = f'_x(x, y) + \lambda g'_x(x, y) &= 0 \\ \Phi'_y(x, y) = f'_y(x, y) + \lambda g'_y(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned}$$

Solving this system we obtain critical points of the function Φ. Then we must verify, for example by means of the second derivatives test, whether these points are points of extrema.

Example 7. Find the greatest and the least values that the function $f(x, y) = xy$ takes on the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$.

Example 8. At which point of the circle $x^2 + y^2 = 1$ does the sum $x + y$ have extrema?

Global (Absolute) Extrema.

In many, especially optimization problems, we are interested in the greatest or the least value of a function on a subset of its domain of definition, in other words, in the global extrema of a function on a set.

Let f be a function of two variables and let a set $M \subset D(f)$. The maximal (minimal) value of f attained on the set M is called the **global maximum (minimum)** of f on M.

If M is a closed bounded set in E_2 and f is a function continuous on M, then the global extrema on M are attained and they can be found in the following three steps.

1. We find local extrema inside the set M (In fact, it is sufficient to find values at all critical interior points of M.)
2. With the aid of constrained extrema we find the greatest and the least values of f on the boundary of M.
3. The greatest (the least) of all found values is the global maximum (minimum) of f on M.

Example 9. Find the global extrema of the function $f(x,y) = y^2 - x^2$ on the set $M = \{[x,y] : x^2 + y^2 \leq 4\}$.

Example 10. Find the global extrema of the function $f(x,y) = 3xy - 6x - 3y - 7$ on the closed triangular region with vertices $[0,0]$, $[3,0]$ and $[0,3]$.

Example 11. Find the global extrema of the function $f(x,y) = x^2 - 2y^2$ on the set M, given by inequalities: $M = \{[x,y] : x \geq 0,\ y \geq 0,\ y \leq -x + 3\}$.

Speaking about global (absolute) extrema, without specifying the set M, it means that $M = D(f)$.

Example 12. Let two rivers be shaped like graphs of the functions $y = x^2$ and $y = x - 2$. Find the length of the shortest canal joining them.

Example 13. Determine dimensions of a rectangular box, open at the top, having a volume of 32 ft^3 and requiring the least amount of material for its construction.