6^{th} Lecture

Continuity, Partial Derivatives of Functions of Two Variables, Total Differentials, Tangent Planes.

Continuity.

Similarly like limit, continuity at a point for functions of several variables is defined in the same way as for functions of one variable.

Let f be a function of n variables, defined in a neighbourhood of a point $A \in E_n$. It is said, that the function f is **continuous** at the point A, if

$$\lim_{X \to A} f(X) = f(A).$$

The following properties of continuous functions are straightforward consequences of limit properties stated above.

Let us suppose, that two functions of n variables f_1 a f_2, both defined in a neighbourhood of a point $A \in E_n$, are continuous at A. Then functions

1. $c_1 f_1 + c_2 f_2$, for any real constants c_1, c_2,
2. $f_1 . f_2$,
3. $\dfrac{f_1}{f_2}$, for $f_2(A) \neq 0$.

are continuous at A. It should be mentioned, that if A is a boundary point of the domain of definition of a function f, then f cannot be continuous at A in a standard sense. This is the reason, why we define a new notion, continuity of a function at a point **with respect to a set**, which is a kind of analogy of one-sided limits in the real case.

Let a function f be defined on a set $M \subset E_n$ and let a point $A \in M$. It is said, that the **function f is continuous at the point A with respect to the set M**, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $X \in N_\delta(A) \cap M$ then $|f(X) - f(A)| < \varepsilon$.

It is said, that a function f is continuous on a set M, if it is continuous at each $X \in M$ with respect to the set M.

Example 1. Show, that the following function of two variables

$$f(x, y) = \begin{cases} 1, & \text{if} \quad y \geq 0 \\ -1, & \text{if} \quad y < 0 \end{cases}$$

is continuous on the sets M_1 and M_2 and it is not continuous on the set M, for $M_1 = \{[x, y] : -1 < y < 0\}$, $M_2 = \{[x, y] : 0 \leq y < 1\}$ and $M = M_1 \cup M_2$.

A function f of several variables, continuous on a closed, bounded and connected set M has on this set properties, analogical to them, of a continuous function of one variable on a closed interval.

1. f is bounded on M,

2. f reaches its minimal and maximal values on M,

3. if A and B are two points from M such that $f(A) \neq f(B)$ and c is a real number between $f(A)$ and $f(B)$, then there exists at least one point $X \in M$ such that $f(X) = c$.

A consequence of these properties is, that any function, continuous on a closed, bounded and connected set, maps this set on a closed interval, or a one-point set.

Partial Derivatives.

Let f be a function of two variables. Let $A = [x_0, y_0]$ be an interior point from its domain of definition. Let as denote by $M_x = \{x : [x, y_0] \in D(f)\} \subset E_1$. Let us define on M_x a new function g of one variable: $g(x) = f(x, y_0)$, for each $x \in M_x$. If there exists derivative $g'(x_0)$, it is called the **partial derivative of f at A, with respect to x** and it is denoted by $f'_x(x_0, y_0)$ or $f'_x(A)$, also $\dfrac{\partial f}{\partial x}(x_0, y_0)$ or $\dfrac{\partial f}{\partial x}(A)$. We prefer the first of them.

Therefore

$$f'_x(x_0, y_0) \overset{def}{=} \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

Similarly we define and denote **partial derivative of f at A with respect to y.**

$$f'_y(x_0, y_0) \overset{def}{=} \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

The Geometric meaning of Partial derivatives.

The real number $f'_x(x_0, y_0)$ is equal to the slope, relative to O_x, of the tangent line to the section of the surface $z = f(x, y)$ by the plane $y = y_0$, drawn through the point $T = [x_0, y_0, f(x_0, y_0)]$. The geometric meaning of $f'_y(x_0, y_0)$ is analogical.

Example 2. Compute $f'_x(A)$ a $f'_y(A)$, if

a) $f(x, y) = 3xy - x^3 - y^3, \quad A = [1, 2]$

b) $f(x, y) = x^y, \quad A = [e, 1]$

c) $f(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}, \quad A = [4, 3]$

Example 3. Show, that the function $f(x, y) = |x| + |y|$ has no partial derivatives at the point $O = [0, 0]$.

It can be easily proved, that the function from the next example is continuous at the point $[0, 0]$, despite of the fact, that partial derivatives at this point do not exist. It is similar to the real case, where continuity at a point does not imply differentiability.

But in contrast to the real case, where differentiability implies continuity at a point, for functions of two variables continuity at a point does not follow from the existence of partial derivatives.

Example 4. Show, that the function

$$f(x, y) = \begin{cases} 0, & \text{if} \quad xy \neq 0 \\ 1, & \text{if} \quad xy = 0 \end{cases}$$

is not continuous at $[0,0]$, nevertheless $f'_x(0,0)$ and $f'_y(0,0)$ exist.

If a function of two variables f has at each point $A \in M \subset E_2$ the partial derivative $f'_x(A)$, then, in fact, we have define on the set M a new function of two variables, which assigns to each point $A \in M$ the value $f'_x(A)$. This function is called **(the first) partial derivative of f with respect to x** and denoted by f'_x or $\dfrac{\partial f}{\partial x}$.

Analogously we define **(the first) partial derivative of f with respect to y** and we denote it by f'_y or $\dfrac{\partial f}{\partial y}$.

Partial Derivatives of Higher Orders.

Suppose, that a function of two variables f on a set M has both partial derivatives f'_x and f'_y. These, as functions of two variables, can again possess partial derivatives, with respect to both variables. If these partial derivatives exist, we call them the **second partial derivatives, or partial derivatives of the second order of the function f**. According to the order of differentiation we obtain four second-order partial derivatives, denoted by one of the following ways (we will prefer the last one).

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f'_x)'_x = f''_{xx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f'_x)'_y = f''_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f'_y)'_x = f''_{yx}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = (f'_y)'_y = f''_{yy}.$$

Example 5. Find all second partial derivatives of the function $f(x,y) = x^2y + x^4y^3$.

The derivatives f''_{xy} and f''_{yx} are called the **mixed** second partial derivatives. For most functions they are equal (like in the last example). Generally, if they are both continuous, then they are identical.

Partial derivatives of the third and higher orders are defined similarly. If second partial derivatives of a function f have partial derivatives, they are called the **third partial derivatives or partial derivatives of the third order of the function f**. If, for example, we differentiate the function f''_{xy} with respect to y, we obtain

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x \partial y^2} = (f''_{xy})'_y = f'''_{xyy}.$$

Example 6. Find partial derivatives of all orders of the function $f(x,y) = 3x^2y - xy^2$.

Total Differentials and Tangent Planes.

Let $A = [x_0, y_0]$ be an interior point from the domain of definition of a function of two variables f. The function f is said to be **differentiable** at the point A if there exist partial derivatives $f'_x(A)$ and $f'_y(A)$ and the increment $f(X) - f(A)$ is expressible in the form

$$f(X) - f(A) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0) + \omega(X).d(X,A),$$

where $d(X, A)$ is the distance of the points A and X and $\omega(X)$ is a function, continuous at A and such that $\omega(A) = 0$.

Remark. It can be simply proved, that the differentiability of f at A implies its continuity at A. On the other hand, differentiability of f at A does not follow from continuity. Sufficient condition of differentiability at a point A is the existence and continuity of partial derivatives at A.

The expression $f'_x(A)(x - x_0) + f'_y(A)(y - y_0)$ is called the **total differential** of the function f at the point $A = [x_0, y_0]$ and it is denoted by $df_A(X)$ or $df_A(x, y)$. Therefore

$$df_A(x, y) \stackrel{def}{=} f'_x(A)(x - x_0) + f'_y(A)(y - y_0)$$

Example 7. If $f(x, y) = x^2y$ and $A = [2, 3]$, find the function $df_A(x, y)$ and its value at the point $[1, 1]$.

Thus the condition of differentiability can be written in the form

$$f(X) - f(A) = df_A(X) + \omega(X).d(X, A).$$

If we neglect the last member in this equality, we obtain an approximate formula, used in numerical mathematics for estimation of values of the function f:

$$f(X) - f(A) \doteq df_A(X) \quad \text{or} \quad f(X) \doteq f(A) + df_A(X)$$

Example 8. By means of the total differential calculate approximately $\sqrt{2.1 \cdot 8.05}$.

Example 9. The volume V of a right circular cylinder is to be calculated from measured values of r (radius of base) and h (height). Suppose, that $r = 2cm$ was measured with an error no more than $0.001cm$ and $h = 4cm$ with an error no more than $0.002cm$. Estimate error in the calculation of V.

From the geometric point of view differentiability of a function f at a point $A = [x_0, y_0]$ guarantees the existence of an unambiguously determined plane, containing two intersecting lines, lying in the planes $x = x_0$ and $y = y_0$, both tangent to $G(f)$ at the point $T = [x_0, y_0, f(x_0, y_0)]$. This plane is called the **tangent plane** to the graph of f at the point T. It can be simply derived that equation of this tangent plane τ is

$$\tau: \quad z - f(x_0, y_0) = f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$

It follows, that the value of the total differential $df_A(X)$ is equal to the corresponding increment of the z-coordinate on the tangent plane.

The straight line passing through the point T and perpendicular to the plane τ is called the **normal** to $G(f)$ at T.

Example 10. Find equations of tangent planes and normals to $G(f)$ at the point T, if
1) $f(x, y) = x^2 + 2y^2, \quad T = [1, 1, ?]$
2) $f(x, y) = x^2, \quad T = [0, 1, 0]$